MT1001 18/19 S1

Tuesday, 12 January 2021 1:38 AM

1. (a) $p' = (1 + 10\%) \cdot p$

$p' = 3.165 + 0.253x, \quad 1 \le x \le 10$

At equilibrium:

$3.465 + 0.253x = \frac{5.03}{x} + 1.35$

$0.253x^2 + 2.115x - 5.03 = 0$

Solved for $x_1 = 1.93, \ x_2 = -10.29$

Given $1 \le x \le 10$

we choose $x_1 = 1.93$ and $p = 3.96$

(b) $p' = (1 + 10\%) \cdot p$

$p' = 1.562 + 0.77e^{0.3x}, \quad 1 \le x \le 10$

At equilibrium:

$1.562 + 0.77e^{0.3x} = \frac{4.5}{x} + 2.54$

$e^{0.3x} - 1.270 - \frac{5.844}{x} = 0$

Using Newton's Iteration:

$f(x) = e^{0.3x} - 1.270 - \frac{5.844}{x} = 0$

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

Starting with $x_1 = 4$:

$x_2 = 4 - \frac{e^{1.2} - 1.270 - \frac{5.844}{4}}{0.3 \cdot e^{1.2} + \frac{5.844}{16}} = 3.56723$

(c) At a price level of \$4,

$x_{supply} = \frac{4 - 3.15}{0.23} = 3.70$

$x_{demand} = 1.90$

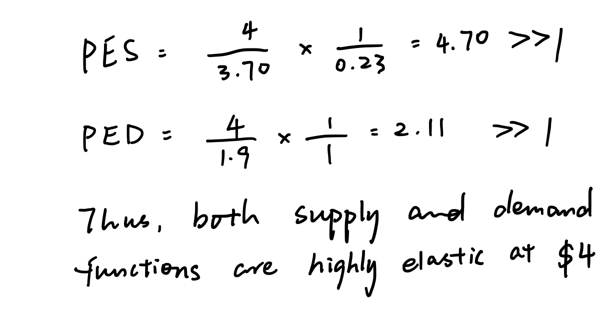
$PES = \frac{4}{3.70} \times \frac{1}{0.23} = 4.70 >> 1$

$PED = \frac{4}{1.9} \times \frac{1}{1} = 2.11 >> 1$

Thus, both supply and demand functions are highly elastic at \$4.

2. (a) Current Balance:

$15{,}050\text{K} - 1{,}000\text{K} = 14{,}050\text{K}$



In the 4th month:

$14{,}050\text{K} \times (1 + \frac{3.5\%}{12})^4 = 14214635$

Thus, by formula

$14214635 = \frac{150000}{\frac{35\%}{12}} \left[1 - (1 + \frac{3.5\%}{12})^{-x}\right]$

Solved for $x \approx 111$

(b) In the 4th month:

$14214635 - \frac{150000}{\frac{3.5\%}{12}} \left[1 - (1 + \frac{3.5\%}{12})^{-111}\right]$

$= Y \div (1 + \frac{3.5\%}{12})^{111}$

Solved for $Y = 11918.5$

3. (a) $\int \frac{\ln x}{(1+x)^2} dx$

$= -\frac{\ln x}{1+x} + \int \frac{1}{x} \cdot \frac{1}{1+x} dx$

$= -\frac{\ln x}{1+x} + \int \frac{A}{x} + \frac{B}{1+x} dx$

(fraction decomposition)

Solved for $\begin{cases} A = 1 \\ B = -1 \end{cases}$

$= -\frac{\ln x}{1+x} + \ln x - \ln(1+x) + C$

(b) $f(x) = \int_1^x \frac{\ln(a^2+1)}{a} 2a \cdot da$

(let $a = \sqrt{t}$, $t = a^2$, $dt = 2a\, da$)

$= 2\int_1^x \ln(a^2+1)\, da$

$= 2\left(a \cdot \ln(a^2+1)\Big|_1^x - \int_1^x \frac{a}{a^2+1} da\right)$

$= 2\left(a \cdot \ln(a^2+1)\Big|_1^x - \frac{1}{2}\int_1^x \frac{dc}{c}\right)$

(let $c = a^2 + 1$, $dc = 2a \cdot da$)

$= 2x\ln(x^2+1) - 2\ln 2 - \frac{1}{2}\ln(t+1)\Big|_1^x$

$= 2x\ln(x^2+1) - \frac{3}{2}\ln 2 - \frac{1}{2}\ln(x+1)$

[I'm not sure about the next step]

(c) $p = 8$

$8 = \frac{10}{\ln(x+2)}$ solved for $x = 1.49$

$CS = \int_0^{1.49} \frac{10}{\ln(x+2)} - 8\, dx$

By Simpson's Rule:

$n = 8, \quad \Delta x = \frac{1.49 - 0}{8} = 0.1863$

$f(x) = \frac{10}{\ln(x+2)} - 8$

Simpson Estimate $= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + \cdots + 4f(x_7) + f(x_8)\right] \approx 0.3645$

4. (a) $f'_x(x,y) = e^{(x+y)} \cdot \left[(y + \frac{x^3}{3}) + x^2\right]$

$f'_y(x,y) = e^{(x+y)} \cdot \left[1 + y + \frac{x^3}{3}\right]$

$\Rightarrow \begin{cases} y + \frac{x^3}{3} = -x^2 \\ y + \frac{x^3}{3} = -1 \end{cases} \Rightarrow (1, -\frac{4}{3})$ and $(-1, -\frac{2}{3})$

$f(1, -\frac{4}{3}) = (-\frac{4}{3} + \frac{1}{3}) \cdot e^{-\frac{1}{3}} = -0.7165$

$\wedge$

$f(-1, -\frac{2}{3}) = (-\frac{2}{3} - \frac{1}{3}) \cdot e^{-\frac{5}{3}} = -0.1888$

Thus, $f(x,y)_{min} = -0.7165$

(b)

① $x = \sqrt{1 - y^2 - z^2}$

$f_y = -2 + \frac{-2y}{2x} = -2 - \frac{y}{x} = 0$

$f_z = 2 - \frac{z}{x} = 0$

$\Rightarrow \begin{cases} y = -2x \\ z = 2x \end{cases}$

② $x = -\sqrt{1 - y^2 - z^2}$

$f_y = -2 + \frac{y}{x} = 0$

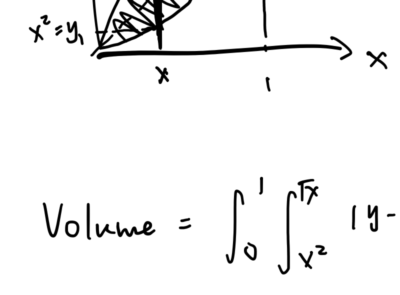
$f_z = 2 + \frac{z}{x} = 0$

$\Rightarrow \begin{cases} y = 2x \\ z = -2x \end{cases}$

Thus, $f_{max} = x - 2y - 2z = x + 0 = x$

Since $x_{max} = 1$, $f_{max} = 1$

(c)



Volume $= \int_0^1 \int_{x^2}^{\sqrt{x}} |y - x|\, dy\, dx$

① $y \ge x$

$V_1 = \int_0^1 \int_x^{\sqrt{x}} y - x\, dy\, dx$

$= \int_0^1 \left(\frac{y^2}{2}\Big|_x^{\sqrt{x}} - xy\Big|_x^{\sqrt{x}}\right) dx$

$= \int_0^1 \frac{x}{2} - \frac{x^2}{2} dx - \int_0^1 x^{\frac{3}{2}} - x^2 dx$

$= \frac{1}{4} - \frac{1}{6} - \left(\frac{2}{5} - \frac{1}{3}\right)$

$= \frac{1}{12} - \frac{1}{15} = \frac{1}{60}$

② $y < x$

$V_2 = \int_0^1 \int_{x^2}^{x} x - y\, dy\, dx$

$= \int_0^1 \left(xy - \frac{y^2}{2}\right)\Big|_{x^2}^{x} dx$

$= \int_0^1 x^2 - \frac{x^2}{2} - x^3 + \frac{x^4}{2} dx$

$= \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$

$= \frac{1}{10} - \frac{1}{12}$

$= \frac{1}{60}$

$\Rightarrow V = V_1 + V_2 = \frac{1}{30}$

All the best ! :)

Shaw