1 (a) (i) required probability $= 40\% \times 0.2 + 30\% \times 0.1 + 20\% \times 0.5 + 30\% \times 0.2$

$= 0.27$

(ii) required probability $= (30\% \times 0.1)/0.27$

$= \frac{1}{9} = 0.111$

1 (b) (i) $\frac{4}{27} + 2p + \frac{2}{27} + p^2 = 1$

$p^2 + 2p - \frac{7}{9} = 0$

$p = \frac{1}{3}$ or $p = -\frac{7}{3}$ (rej. since $p > 0$)

(ii)

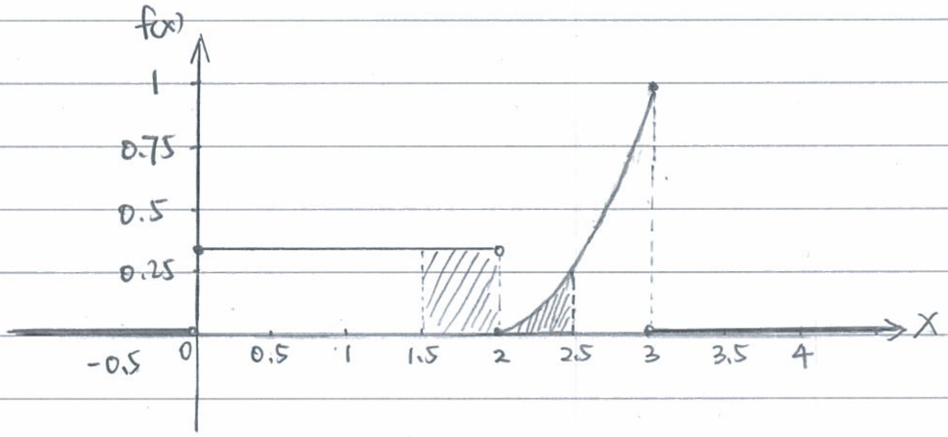
x	0	1	1.5	3	Total
$f(x)$	$\frac{4}{27}$	$\frac{2}{3}$	$\frac{2}{27}$	$\frac{1}{9}$	1
$xf(x)$	0	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{10}{9} = 1.11$
$x^2 f(x)$	0	$\frac{2}{3}$	$\frac{1}{6}$	1	$\frac{11}{6} = 1.83$

$E(X) = \sum_x x \cdot f(x) = 1.11$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{11}{6} - (\frac{10}{9})^2 = \frac{97}{162} = 0.60$ (to 2 d.p.)

2 (a)



(b) $F(x) = \int_{-\infty}^{x} \frac{1}{3}\, dt$ for $x \le 2$

$F(x) = \int_0^x \frac{1}{3}\, dt$, for $0 \le x \le 2$; $F(x) = 0$, for $x < 0$

since $f(x) = 0$ when $x < 0$

$= \frac{1}{3}x$, $0 \le x \le 2$

$\therefore F(2) = (\frac{1}{3})(2) = \frac{2}{3}$

2 (c) $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$$= \begin{cases} 0, & x < 0 \\ \frac{x}{3}, & 0 \le x < 2 \\ \frac{2}{3} + \int_2^x (t-2)^2\, dt, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

$$\int_2^x (t-2)^2\, dt = \int_2^x t^2 - 4t + 4\, dt$$
$$= \frac{x^3}{3} - 2x^2 + 4x - \frac{8}{3}$$

$$\Rightarrow \frac{2}{3} + \int_2^x (t-2)^2\, dt = \frac{x^3}{3} - 2x^2 + 4x - 2$$

$$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{3}, & 0 \le x < 2 \\ \frac{x^3}{3} - 2x^2 + 4x - 2, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

2 (d) $E(X \cdot Y) = E(X) \cdot E(Y)$, since X & Y are independent.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$
$$= \int_0^2 \frac{1}{3} x\, dx + \int_2^3 x \cdot (x-2)^2\, dx$$
$$= \frac{2}{3} + \frac{11}{12}$$
$$= \frac{19}{12}$$

$$E(Y) = \int_0^2 \frac{1}{2}\, dx = 1$$

$$\therefore E(XY) = E(X) \cdot E(Y) = \frac{19}{12}.$$

3(a)(i) $\lambda = 1.25$, $t = 4$

Let X represent number of flaws in a 4 mm portion of wire.

$\mu = \lambda t = 1.25 \times 4 = 5$

$P(X = x) = p_o(x; 5)$

$P(X=0) = e^{-5} = 0.00673795$

(a)(ii) Let Y represent number of portions with no flaws out of 1000 portions inspected.

$P(Y=y) = b(y; n, p) = b(y; 1000, e^{-5})$

using Normal approximation,

$\mu = np = 1000 \times e^{-5} = 6.73795$

$\sigma^2 = npq = 1000 \times e^{-5} \times (1 - e^{-5}) = 6.69255$

$P(Y \le 10) = P(Y < 10.5)$ by continuity correction.

$P\left(Z < \frac{10.5 - 6.73795}{\sqrt{6.69255}}\right)$

$= P(Z < 1.45)$

$= 0.9265$

3(b) $\mu = 61.4$, $\sigma^2 = 16 \Rightarrow \sigma = 4$

$P(40 < X \le 70 \mid X > 50)$

$= \frac{P(50 < X \le 70)}{P(X > 50)}$

$= \frac{0.982036}{0.997814}$ (from G.C.)

$= 0.9842.$

4(a) $n = 36$, $\bar{X} = 72.5$, $S_x^2 = 16 \Rightarrow S = 4$

σ^2 unknown, $n = 36 \geq 30 \Rightarrow Z = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

$$\bar{x} - z_{\frac{\alpha}{2}}\left(\frac{s}{\sqrt{n}}\right) < \mu < \bar{x} + z_{\frac{\alpha}{2}}\left(\frac{s}{\sqrt{n}}\right)$$

At 95% confidence interval, $z_{\frac{\alpha}{2}} = z_{0.025} = 1.960$

$$72.5 - 1.960 \times \left(\frac{4}{\sqrt{36}}\right) < \mu < 72.5 + 1.960 \times \frac{4}{\sqrt{36}}$$

$$71.19 < \mu < 73.81 \quad (\text{to } 2 \text{ d.p.})$$

(b) $E(X_0 - \bar{X}) = 0$, $Var(X_0 - \bar{X}) = s^2 + \frac{s^2}{n}$

$$Z = \frac{X_0 - \bar{X}}{s\sqrt{1+\frac{1}{n}}}$$

$$\bar{x} - z_{\frac{\alpha}{2}}(s)\sqrt{1+\frac{1}{n}} < x_0 < \bar{x} + z_{\frac{\alpha}{2}}(s)\sqrt{1+\frac{1}{n}}$$

At 95% prediction level, $z_{\frac{\alpha}{2}} = z_{0.025} = 1.960$

$$72.5 - 1.960 \times 4 \times \sqrt{1+\frac{1}{36}} < x_0 < 72.5 + 1.960 \times 4 \times \sqrt{1+\frac{1}{36}}$$

$$64.55 < x_0 < 80.45 \quad (\text{to } 2\text{d.p.})$$

(c) Error $= |\bar{x} - \mu|$

95% confident that error < 1, then n satisfies:

$$n > \left(\frac{z_{\frac{\alpha}{2}} \cdot s}{\varepsilon}\right)^2$$

$$n > \left(\frac{1.960 \times 4}{1}\right)^2$$

$$n > 61.4656$$

$\therefore$ sample size required is at least 62.

(d) $n_Y = 32$, $\bar{Y} = 70.3$, $S_Y^2 = 18 \Rightarrow S_Y = 3\sqrt{2}$

Both n_X & n_Y are greater than 30 $\Rightarrow Z = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{S_X^2/n_X + S_Y^2/n_Y}}$ by CLT

$P(\bar{X} - \bar{Y} < 3)$

$= P\left(Z < \frac{3 - (72.5 - 70.3)}{\sqrt{16/36 + 18/32}}\right)$

$= P(Z < 0.79724)$

$= 0.7852$

5(a) $H_0: \mu = 75$

$H_1: \mu > 75$

$\alpha = 0.01$

Test statistic: $Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ (σ^2 unknown, $n = 50 > 30$)

critical region: $Z > 2.328$

$\bar{x} = 78.4 \quad s = 5.8 \quad n = 50$

$z = (78.4 - 75)/(5.8/\sqrt{50}) = 4.145 > 2.328$ in the critical region.

$\therefore$ Reject H_0 and conclude that the average tensile strength is at least 75 kg.

5(b)(i) $S_{xx} = \sum x^2 - n(\bar{x})^2$

$= \sum x^2 - n\left(\frac{\sum x}{n}\right)^2$

$= 100 - 10\left(\frac{30}{10}\right)^2 = 10$

$S_{xy} = \sum xy - n\bar{x}\bar{y} = \sum xy - n\left(\frac{\sum x}{n}\right)\left(\frac{\sum y}{n}\right)$

$= 170 - 10\left(\frac{30}{10}\right)\left(\frac{50}{10}\right) = 20$

(ii) $b_1 = \frac{S_{xy}}{S_{xx}} = \frac{20}{10} = 2$

$b_0 = \bar{y} - b_1\bar{x} = 5 - 2 \times 3 = -1$

$\therefore \hat{y} = -1 + 2x$

when $x = 5.8$, $\hat{y} = -1 + 5.8 \times 2 = 10.6$

(iii) $s^2 = \frac{SSE}{n-2} = \frac{S_{yy} - bS_{xy}}{n-2} = \frac{300 - 10\left(\frac{50}{10}\right)^2 - 2 \times 20}{10 - 2} = \frac{10}{8} = 1.25$

At 90% confidence level, $t_{\frac{\alpha}{2}} = t_{0.05} = 1.860$, $v = 10 - 2 = 8$

$b_1 - t_{\frac{\alpha}{2}} \cdot s/\sqrt{S_{xx}} < \beta_1 < b_1 + t_{\frac{\alpha}{2}} \cdot s/\sqrt{S_{xx}}$

$2 - 1.860 \times \frac{\sqrt{1.25}}{\sqrt{10}} < \beta_1 < 2 + 1.860 \times \frac{\sqrt{1.25}}{\sqrt{10}}$

$1.3424 < \beta_1 < 2.6576$

$1.34 < \beta_1 < 2.66$ (to 2 d.p.)

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