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2018 13/14 Sem 1

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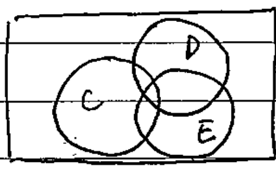
1. (a) (i) $P(A) = \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{1}{10}$

(ii) $P(B) = \frac{2}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{5}$

(iii) Events A and B are not statistically independent.

$\because P(A \cap B) = 0.\quad P(A)P(B) = \frac{1}{50}.\qquad P(A \cap B) \neq P(A)P(B)$

(b) $P(C|DE) = \frac{P(C \cap DE)}{P(DE)} = 0.2$



$P(\overline{D \cap E}) = 0.7 \Rightarrow P(D \cap E) = 0.3$

$P(C \cap DE) = 0.2 \times 0.3 = 0.06$

(i) $P(C \cup DE) = P(C) + P(DE) - P(C \cap DE)$

$= 0.67 + 0.3 - 0.06$

$= 0.91$

(ii) $P(\overline{C \cap D} \cup \overline{E}) = P(\overline{C \cap D \cap E}) = 1 - P(C \cap D \cap E) = 1 - 0.06 = 0.94$

(iii) $P(DE|\overline{C}) = \frac{P(DE \cap \overline{C})}{P(\overline{C})} = \frac{P(DE) - P(C \cap D \cap E)}{1 - P(C)}$

$= \frac{0.3 - 0.06}{1 - 0.67} = \frac{0.24}{0.33} = 0.727$

(iv) $P(C|\overline{D \cap E}) = \frac{P(C \cap \overline{D \cap E})}{P(\overline{D \cap E})} = \frac{P(C) - P(C \cap DE)}{P(\overline{DE})}$

$= \frac{0.67 - 0.06}{0.7} = \frac{0.61}{0.7} = 0.871$

(v) $P(D \cap E) \neq 0$, $\therefore$ Events D and E are not mutually exclusive.

2. (a) $Z = \frac{X - \mu}{\sigma} = \frac{X - 12.975}{0.025}$

$X = 13.025,\quad Z = \frac{13.025 - 12.975}{0.025} = 2$

$P(X \geq 13.025) = 1 - \Phi(Z) = 1 - 0.9772 = 0.0228$

(b) $P(\text{at least 5 bearings acceptable}) = {}_6C_5\,(1-0.160)^5 \times 0.160 + {}_6C_6 \times (1-0.160)^6$

$= 6 \times 0.84^5 \times 0.16 + 0.84^6$

$= 0.753$

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(c) $\delta = 0.045$ $\quad 12.955 \le X \le 13.045$

$Z_1 = \frac{12.955 - 12.975}{0.025} = -0.8$ $\quad \Phi(Z_1) = 1 - \Phi(0.8) = 0.212$

$Z_2 = \frac{13.045 - 12.975}{0.025} = 2.8$ $\quad \Phi(Z_2) = 0.997$

$P(\text{unacceptable}) = \Phi(Z_1) + 1 - \Phi(Z_2)$

$= 0.212 + 1 - 0.997$

$= 0.215$

(d) ~~$P(X \ge 13) = \Phi(Z$~~

$Z_3 = \frac{13 - 12.975}{0.025} = 1.$ $\quad \Phi(Z_3) = 0.8413$

$P(X \ge 13 \mid 12.955 \le X \le 13.045) = \frac{P(13 \le X \le 13.045)}{P(12.955 \le X \le 13.045)}$

$= \frac{\Phi(Z_2) - \Phi(Z_3)}{\Phi(Z_2) - \Phi(Z_1)} = \frac{0.9974 - 0.8413}{0.9974 - 0.212} = \frac{0.1561}{0.7854} = 0.1988$

(e) $P(D < 13.000 + \delta) = 0.9987 \Rightarrow Z_4 = 3.00$

$\frac{X - 12.975}{0.025} = 3.00.$ $\quad X = 13.050$

$\delta = 13.050 - 13.000 = 0.050$ mm

3. (a) (i) $E[Y] = E[\alpha \sum_{i=1}^{n} a_i X_i]$

$= \alpha E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n]$

$= \alpha(a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n])$

$= \alpha(a_1 \mu + a_2 \mu + \cdots + a_n \mu)$

$= \alpha \mu \sum_{i=1}^{n} a_i = \mu$ when Y is unbiased

$\therefore \alpha = \frac{1}{\sum_{i=1}^{n} a_i}$

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(ii) $var[Y] = var[\alpha \sum_{i=1}^{n} a_i X_i]$

$= \alpha^2 (\sum_{i=1}^{n} var[a_i X_i])$

$= \alpha^2 \sum_{i=1}^{n} a_i^2 \, var[X_i]$

$= \alpha^2 \sum_{i=1}^{n} a_i^2 \cdot \sigma^2$

When Y is ~~con~~ unbiased, $\alpha = \frac{1}{\sum_{i=1}^{n} a_i}$

$var[Y] = \frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2} \times \sigma^2 = \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 + k} \times \sigma^2$. k is some positive number

As $n \to \infty$, $k \to \infty$. $\lim_{n\to\infty} var[Y] = \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 + k} \sigma^2 = 0$

$\therefore$ Y is consistent when it is unbiased.

(iii) $E[W] = E[\beta \sum_{i=1}^{n} a_i (X_i - Y_e)^2]$

$= \beta E[\sum_{i=1}^{n} a_i (X_i^2 - 2X_i Y_e + Y_e^2)]$

$= \beta \sum_{i=1}^{n} a_i (E[X_i^2] - 2E[X_i Y_e] + E[Y_e^2])$ $Y_e = Y$.

$(\because X$ and Y are independent$)$ $= \beta \sum_{i=1}^{n} a_i (E[X_i]^2 + var[X_i] - 2E[X_i]E[Y] + E[Y]^2 + var[Y])$

$= \beta \sum_{i=1}^{n} a_i (\mu^2 + \sigma^2 - 2\cdot\mu\cdot\mu + \mu^2 + \frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2} \sigma^2)$

$= \beta \sum_{i=1}^{n} a_i (\frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2} + 1) \sigma^2$

When W is unbiased, $E[W] = \sigma^2$.

$\Rightarrow \beta \sum_{i=1}^{n} (a_i + \frac{a_i \sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2}) \sigma^2 = \sigma^2$

$\beta = \frac{1}{\sum_{i=1}^{n} (a_i + \frac{a_i \sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2})}$

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t distribution since $n<30$ ~~σ unknown~~

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(b) (i) $T = \frac{\bar{X} - \mu}{\sqrt{S^2/n}} = \frac{72 - \mu}{\sqrt{16^2/16}} = \frac{72 - \mu}{4} \sim t_{0.025, 15}$

$72 - 2.131 \times 4 \le \mu \le 72 + 2.131 \times 4$

$63.476 \le \mu \le 80.524$

(ii) $V = \frac{15 \times 16^2}{\sigma^2} \sim \chi^2_{15}$

$15 \times 16^2 / \chi^2_{0.025, 15} \le \sigma^2 \le 15 \times 16^2 / \chi^2_{0.975, 15}$

$15 \times 16^2 / 27.488 \le \sigma^2 \le 15 \times 16^2 / 6.262$

$139.70 \le \sigma^2 \le 613.22$

(iii) $T = \frac{X_{17} - \bar{X}}{S\sqrt{1 + 1/n}} = \frac{X_{17} - 72}{16\sqrt{1+1/16}} \sim t_{0.025, 15}$

$X_{17} = 72 \pm 2.131 \times 16 \sqrt{1 + 1/16} = 72 \pm 35.145$

(iv) $\cancel{T_{17} = \frac{\bar{X}_{17} - \mu}{\sqrt{S^2/n}}}$

The time needed for 18^{th} and 17^{th} worker ~~are~~ is independent

$var[X_{17} - X_{18}] = var[X_{18}] + var[X_{17}] = 2S^2.$

$T = \frac{(X_{17} - X_{18}) - (\bar{X} - \bar{X})}{\sqrt{2S^2}} = \frac{X_{17} - X_{18}}{\sqrt{2} \times 16} = \frac{16}{\sqrt{2} \times 16} = 0.707$

$\because$ $t_{0.25, 15} = 0.691$ $t_{0.10, 15} = 1.341$

$\frac{0.707 - 0.691}{1.341 - 0.691} = \frac{P - 0.25}{0.10 - 0.25}$ $P = 0.246$

4. (a) Test $H_0: \mu_B - 1.05\mu_A \le 0$ against $H_1: \mu_B - 1.05\mu_A > 0$ at 90% confidance interval

$$T_A = \frac{\bar{X}_A - \mu_A}{S_A/\sqrt{n}} = \frac{1.0 - \mu_A}{0.1/\sqrt{8}}$$

$$T_B = \frac{\bar{X}_B - \mu_B}{S_B/\sqrt{n}} = \frac{1.1 - \mu_B}{0.1/\sqrt{8}}$$

$$T = \frac{\bar{X}_B - 1.05\bar{X}_A - (\mu_B - 1.05\mu_A)}{\sqrt{0.1^2/8 + 1.05^2 \times 0.1^2/8}} = \frac{1.1 - 1.05 - (\mu_B - 1.05\mu_A)}{\sqrt{0.1^2/8 + 1.05^2 \times 0.1^2/8}}$$

$$= \frac{0.05 - 0}{0.0513} = 0.975 < t_{0.10,7} = 1.415$$

We cannot reject H_0. No conclusion.

(b) (i) $K = cq^d \Rightarrow \ln K = \ln c + d\ln q$

$X = \ln q$	$Y = \ln K$	X^2	Y^2	XY
1.0986	−0.9365	1.2069	0.8770	−1.0288
1.3863	−0.6911	1.9218	0.4777	−0.9581
1.7916	−0.2033	3.2104	0.04135	−0.3643
1.9459	0.08158	3.7866	0.006655	0.1587

$\bar{X} = 1.5556$ $\bar{Y} = -0.4374$ $\Sigma X^2 = 10.1257$ $\Sigma Y^2 = 1.4027$ $\Sigma XY = -2.1926$

$$\hat{a}_1 = \frac{\sum_{k=1}^{4} X_k Y_k - 4\bar{X}\bar{Y}}{\sum_{k=1}^{4} X_k^2 - 4\bar{X}^2} = \frac{\overset{-2.1926}{\cancel{-0.5481}} - 4 \times 1.5556 \times (-0.4374)}{\underset{10.1257}{\cancel{2.5314}} - 4 \times 1.5556^2} = 1.186$$

$\Rightarrow d = \hat{a}_1 = 1.186$

$\hat{a}_0 = \bar{Y} - \hat{a}_1\bar{X} = -0.4374 - 1.186 \times 1.5556 = -2.28$

$\Rightarrow c = e^{\hat{a}_0} = 0.102$

(ii) $SS_E = S_{YY} - \hat{a}_1 S_{XY} = [1.4027 - 4 \times (-0.4374)^2] - 1.186 \times [-2.1926 - 4 \times 1.5556 \times (-0.4374)]$

$= 0.6376 - 1.186 \times 0.5289 = 0.009906$

$$d = \hat{a}_1 \pm t_{0.025,2}\sqrt{\frac{SS_E}{2 \times S_{XX}}} = 1.186 \pm 4.303 \times \sqrt{\frac{0.009906}{2 \times (10.1257 - 4 \times 1.5556^2)}}$$

$= 1.186 \pm 0.4537$