NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2009-2010

CV2001 – Engineering Probability and Statistics

MT2301 – Statistics

November - December 2009 **Time Allowed: 2½ hours**

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **THREE (3)** pages.
2. Answer **ALL FOUR (4)** questions.
3. An Appendix of **FIVE (5)** pages is attached together with this paper.
4. All questions carry equal marks.

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1. (a) The probability that a student owns a computer, $P(C)$ is 0.7, that a student owns a hand-phone, $P(H)$ is 0.8, and that a student owns both a computer and a hand-phone, $P(C \cap H)$ is 0.6.

 (i) Find the probability that a student owns either a computer or a hand-phone or both.

 (ii) Find the probability that a student owns either a computer or a hand-phone, but not both.

 (iii) Find $P(\overline{C} \cap \overline{H})$.

 (iv) Find $P(C \cup H \mid \overline{H})$.

 (v) Are event C and event H statistically independent? Justify your answer.

(15 marks)

 (b) Find the value of the expectation, $E\,[(3Z + 4)^2]$, where Z is the standard Normal random variable.

(5 marks)

Note: Question No.1 continues on page 2

(c) Given two Binomial random variables:

X : number of successes among n trials, with probability of success at each trial = p; and

Y : number of successes among n trials, with probability of success at each trial = $1-p$.

If $P(X=x)=P(Y=y)$, find y in terms of n and x.

[Hint: $P(X=x)=\frac{n!}{(n-x)!\,x!}p^x(1-p)^{n-x}$]

(5 marks)

2. In a city, the storm runoff can be modelled by a random variable X with the following probability density function

$$f(x)=\frac{x}{6}-\frac{x^2}{36}; \qquad 0\le x\le 6$$
$$=0; \qquad \textit{otherwise}$$

(a) Find the distribution function, $F(X)$ of X.

(5 marks)

(b) Find
(i) the mean μ of X, and
(ii) the variance σ^2 of X.

(10 marks)

(c) The runoff is currently carried by a pipe with a capacity of 3. Overflow will occur when the runoff exceeds the pipe capacity. If overflow occurs after storm, what is the conditional probability that the runoff in this storm is less than 4?

(5 marks)

(d) An engineer considers replacing the current pipe by a larger pipe having a capacity of 4. Suppose there is a probability of 0.7 that the replacement would be completed prior to the next storm. What is the probability of overflow in the next storm?

(Do all your calculations to 4 decimal places)

(5 marks)

3. Use random variable X to represent the public response to a policy, i.e., $X = 1$ means support and $X = 0$ for other cases. The support ratio, p, is defined as $p = P(X = 1)$. A survey with n participants has been carried out, and the result for the ith participant is

$$X_i = \begin{cases} 1 & \text{(when the ith paticipant supports the policy)} \\ 0 & \text{(when the ith paticipant don't support the policy)} \end{cases}, \ (i = 1, 2, \cdots, n).$$

(a) Given that an estimator for the support ratio, p, is $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$, show that $E[\hat{p}] = p$ and $\text{var}[\hat{p}] = \frac{p(1-p)}{n}$.

(5 marks)

(b) If the survey shows that 55 out of 100 participants support the policy, give out the unbiased and consistent estimates for $E[X]$ and $\text{var}[X]$, respectively.

(5 marks)

(c) Consider an estimator for $\text{var}[X]$, $\hat{\sigma}_X^2 = \hat{p}(1-\hat{p}) = \overline{X}(1-\overline{X})$. Is the estimator unbiased or not? If not, please construct an unbiased one accordingly.

(8 marks)

(d) If the survey shows that 55 out of 100 participants support the policy, give out the 95%-confidence interval for the support ratio, p.

(7 marks)

4. (a) The diameter of a batch of components is to be 100 mm and the standard deviation is strictly required to be less than 2 mm. The measured diameters of 10 randomly selected components from the batch are 101.0 mm, 101.3 mm, 100.4 mm, 99.8 mm, 98.8 mm, 100.7 mm, 99.3 mm, 100.8 mm, 98.2 mm and 99.5 mm, respectively. Is this batch acceptable with confidence level 95%? The diameter of these components is assumed to be normally distributed.

(12 marks)

(b) (i) Estimate the coefficients, a_0 and a_1, in the linear regression model $Y = a_0 + a_1 X$ from the data tabulated in Table Q4.

(ii) Give out 90%-confidence interval for the coefficient a_0.

(13 marks)

Table Q4

X	3	4	5	6	7
Y	20.1	24.6	30.3	36	39.4

END OF PAPER

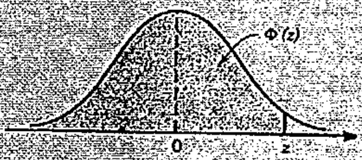
Percentage Points of Standard Normal Distribution

Values of z for given values of $\Phi(z)$ and $D(z) = \Phi(z) - \Phi(-z)$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$
1	−2.326	0.013	41	−0.228	0.539	81	0.878	1.311
2	−2.054	0.025	42	−0.202	0.553	82	0.915	1.341
3	−1.881	0.038	43	−0.176	0.568	83	0.954	1.372
4	−1.751	0.050	44	−0.151	0.583	84	0.994	1.405
5	−1.645	0.063	45	−0.126	0.598	85	1.036	1.440
6	−1.555	0.075	46	−0.100	0.613	86	1.080	1.476
7	−1.476	0.088	47	−0.075	0.628	87	1.126	1.514
8	−1.405	0.100	48	−0.050	0.643	88	1.175	1.555
9	−1.341	0.113	49	−0.025	0.659	89	1.227	1.598
10	−1.282	0.126	50	0.000	0.674	90	1.282	1.645
11	−1.227	0.138	51	0.025	0.690	91	1.341	1.695
12	−1.175	0.151	52	0.050	0.706	92	1.405	1.751
13	−1.126	0.164	53	0.075	0.722	93	1.476	1.812
14	−1.080	0.176	54	0.100	0.739	94	1.555	1.881
15	−1.036	0.189	55	0.126	0.755	95	1.645	1.960
16	−0.994	0.202	56	0.151	0.772	96	1.751	2.054
17	−0.954	0.215	57	0.176	0.789	97	1.881	2.170
18	−0.915	0.228	58	0.202	0.806	97.5	1.960	2.241
19	−0.878	0.240	59	0.228	0.824	98	2.054	2.326
20	−0.842	0.253	60	0.253	0.842	99	2.326	2.576
21	−0.806	0.266	61	0.279	0.860	99.1	2.366	2.612
22	−0.772	0.279	62	0.305	0.878	99.2	2.409	2.652
23	−0.739	0.292	63	0.332	0.896	99.3	2.457	2.697
24	−0.706	0.305	64	0.358	0.915	99.4	2.512	2.748
25	−0.674	0.319	65	0.385	0.935	99.5	2.576	2.807
26	−0.643	0.332	66	0.412	0.954	99.6	2.652	2.878
27	−0.613	0.345	67	0.440	0.974	99.7	2.748	2.968
28	−0.583	0.358	68	0.468	0.994	99.8	2.878	3.090
29	−0.553	0.372	69	0.496	1.015	99.9	3.090	3.291
30	−0.524	0.385	70	0.524	1.036			
31	−0.496	0.399	71	0.553	1.058	99.91	3.121	3.320
32	−0.468	0.412	72	0.583	1.080	99.92	3.156	3.353
33	−0.440	0.426	73	0.613	1.103	99.93	3.195	3.390
34	−0.412	0.440	74	0.643	1.126	99.94	3.239	3.432
35	−0.385	0.454	75	0.674	1.150	99.95	3.291	3.481
36	−0.358	0.468	76	0.706	1.175	99.96	3.353	3.540
37	−0.332	0.482	77	0.739	1.200	99.97	3.432	3.615
38	−0.305	0.496	78	0.772	1.227	99.98	3.540	3.719
39	−0.279	0.510	79	0.806	1.254	99.99	3.719	3.891
40	−0.253	0.524	80	0.842	1.282			

Cumulative Standard Normal Distribution

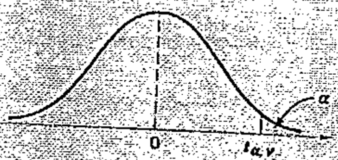
$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$



Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Percentage Points of t-Distribution (v = degree of freedom)

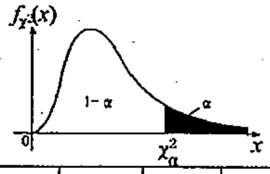


Percentage Points $t_{\alpha,\nu}$ of the t-Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Percentage Points of χ^2-Distribution (n = degree of freedom)

$P\{\chi^2 < \chi^2_\alpha(n)\} = 1 - \alpha$



χ^2_α \ α / n	0.995	0.99	0.975	0.95	0.90	0.75	0.25	0.10	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	0.016	0.102	1.323	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	0.575	2.773	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.581	1.213	4.108	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	1.923	5.385	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	2.675	6.626	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	3.455	7.841	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	4.255	9.037	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	5.071	10.219	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	11.389	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	12.549	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	13.701	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	14.845	18.549	21.026	23.337	26.217	28.299
13	3.565	4.107	5.009	5.892	7.042	9.299	15.984	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	17.117	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.037	18.245	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	19.369	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	20.489	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	21.605	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	22.718	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	23.828	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	24.935	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	17.240	26.039	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	18.137	27.141	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	28.241	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	19.939	29.339	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	30.435	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	31.528	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	32.620	37.916	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	23.567	33.711	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	24.478	34.800	40.256	43.773	46.979	50.892	53.672
31	14.458	15.655	17.539	19.281	21.434	25.390	35.887	41.422	44.985	48.232	52.191	55.003
32	15.134	16.362	18.291	20.072	22.271	26.304	36.973	42.585	46.194	49.480	53.486	56.328
33	15.815	17.074	19.047	20.867	23.110	27.219	38.058	43.745	47.400	50.725	54.776	57.648
34	16.501	17.789	19.806	21.664	23.952	28.136	39.141	44.903	48.602	51.966	56.061	58.964
35	17.192	18.509	20.569	22.465	24.797	29.054	40.223	46.059	49.802	53.203	57.342	60.275
36	17.887	19.233	21.336	23.269	25.643	29.973	41.304	47.212	50.998	54.437	58.619	61.581
37	18.586	19.960	22.106	24.075	26.492	30.893	42.383	48.363	52.192	55.668	59.892	62.883
38	19.289	20.691	22.878	24.884	27.343	31.815	43.462	49.513	53.384	56.896	61.162	64.181
39	19.996	21.426	23.654	25.695	28.196	32.737	44.539	50.660	54.572	58.120	62.428	65.476
40	20.707	22.164	24.433	26.509	29.051	33.660	45.616	51.805	55.758	59.342	63.691	66.766
41	21.421	22.906	25.215	27.326	29.907	34.585	46.692	52.949	56.942	60.561	64.950	68.053
42	22.138	23.650	25.999	28.144	30.765	35.510	47.766	54.090	58.124	61.777	66.206	69.336
43	22.859	24.398	26.785	28.965	31.625	36.436	48.840	55.230	59.304	62.990	67.459	70.616
44	23.584	25.148	27.575	29.787	32.487	37.363	49.913	56.369	60.481	64.201	68.710	71.893
45	24.311	25.901	28.366	30.612	33.350	38.291	50.985	57.505	61.656	65.410	69.957	73.166

Formulae

	Normal Population	Linear Regression
Parameters	μ and σ^2	a_0, a_1 and σ^2
Estimators	$\hat{\mu} = \overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$ $\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(X_k - \overline{X}\right)^2$	$\hat{a}_1 = \frac{S_{XY}}{S_{XX}}$, $\hat{a}_0 = \overline{Y} - \hat{a}_1\overline{X}$ $\hat{\sigma}^2 = \frac{SS_E}{n-2} = \frac{S_{YY} - \hat{a}_1 S_{XY}}{n-2} = \frac{1}{n-2}\sum_{k=1}^{n}\left(Y_k - \hat{Y}_k\right)^2$
	$E[\hat{\mu}] = \mu$ $\text{var}[\hat{\mu}] \equiv \sigma_{\hat{\mu}}^2 = \frac{\sigma^2}{n}$	$E[\hat{a}_1] = a_1$, $E[\hat{a}_0] = a_0$ $\text{var}[\hat{a}_1] \equiv \sigma_{\hat{a}_1}^2 = \frac{\sigma^2}{S_{XX}}$, $\text{var}[\hat{a}_0] \equiv \sigma_{\hat{a}_0}^2 = \sigma^2\left(\frac{1}{n} + \frac{\overline{X}^2}{S_{XX}}\right)$
	$E[\hat{\sigma}^2] = E[S^2] = \sigma^2$	$E[\hat{\sigma}^2] = E\left[\frac{SS_E}{n-2}\right] = \sigma^2$
Distribution	$\frac{\hat{\mu} - E[\hat{\mu}]}{\sigma_{\hat{\mu}}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$	$\frac{\hat{a}_1 - E[\hat{a}_1]}{\sigma_{\hat{a}_1}} = \frac{\hat{a}_1 - a_1}{\sigma/\sqrt{S_{XX}}} \sim N(0,1)$ $\frac{\hat{a}_0 - E[\hat{a}_0]}{\sigma_{\hat{a}_0}} = \frac{\hat{a}_0 - a_0}{\sigma\sqrt{1/n + \overline{X}^2/S_{XX}}} \sim N(0,1)$
	$\frac{\hat{\mu} - E[\hat{\mu}]}{\hat{\sigma}_{\hat{\mu}}} = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$	$\frac{\hat{a}_1 - E[\hat{a}_1]}{\hat{\sigma}_{\hat{a}_1}} = \frac{\hat{a}_1 - a_1}{\hat{\sigma}/\sqrt{S_{XX}}} \sim t_{n-2}$ $\frac{\hat{a}_0 - E[\hat{a}_0]}{\hat{\sigma}_{\hat{a}_0}} = \frac{\hat{a}_0 - a_0}{\hat{\sigma}\sqrt{1/n + \overline{X}^2/S_{XX}}} \sim t_{n-2}$
	$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$	$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} = \frac{SS_E}{\sigma^2} \sim \chi_{n-2}^2$
	$\frac{X_{n+1} - \overline{X}}{\sigma\sqrt{1 + 1/n}} \sim N(0,1)$	$\frac{Y_{n+1} - \hat{Y}_{n+1}}{\sigma\sqrt{1 + 1/n + \left(X_{n+1} - \overline{X}\right)^2/S_{XX}}} \sim N(0,1)$
	$\frac{X_{n+1} - \overline{X}}{\hat{\sigma}\sqrt{1 + 1/n}} \sim t_{n-1}$	$\frac{Y_{n+1} - \hat{Y}_{n+1}}{\hat{\sigma}\sqrt{1 + 1/n + \left(X_{n+1} - \overline{X}\right)^2/S_{XX}}} \sim t_{n-2}$

NANYANG TECHNOLOGICAL UNIVERSITY

CV2001 – Engineering Probability and Statistics   Date   No. 1

Semester I Examination 2009 - 2010

1. (a). $P(C) = 0.7$

$P(H) = 0.8$

$P(C \cap H) = 0.6$

(i) Probability that a student owns either a computer or a hand-phone or both

$= P(C \cup H) = P(C) + P(H) - P(C \cap H)$

$= 0.9$ //

(ii) Probability that a student owns either a computer or a hand-phone, but not both

$= P(C \cup H) - P(C \cap H)$

$= 0.3$ //

(iii) $P(\bar{C} \cap \bar{H}) = 1 - P(\overline{\bar{C} \cap \bar{H}})$

$= 1 - P(C \cup H)$

$= 0.1$ //

(iv) $P(C \cup H | \bar{H}) = \dfrac{P((C \cup H) \cap \bar{H})}{P(\bar{H})}$

$= \dfrac{P(C \cap \bar{H})}{1 - P(H)}$

$= \dfrac{1 - P(H) - P(\bar{C} \cap \bar{H})}{1 - P(H)}$ ← $P(C \cap \bar{H}) + P(\bar{C} \cap \bar{H}) = P(\bar{H}) = 1 - P(H)$

$= 0.5$ //

(v) $P(C) \times P(H)$ ... $P(C \cap H)$

$0.7 \times 0.8 \neq 0.6$

→ So: event C and event H are not statistically independent. //

(b). $Z \sim N(0,1)$

$E[(3Z+4)^2] = E[9Z^2 + 24Z + 16]$

$= 9E[Z^2] + 24E[Z] + 16$

$= 9(\mu^2 + \sigma^2) + 24\mu + 16$

$= 25$ //

1. (c). X : number of success among n trials, with probability of success at each trial = p

Y : number of success among n trials, with probability of success at each trial = 1-p

$$P(X=x) = P(Y=y)$$

$$\frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \frac{n!}{(n-y)!y!} (1-p)^y p^{n-y}$$

$$p^{x+y-n} = \frac{(n-x)!x!}{(n-y)!y!} (1-p)^{y+x-n}$$

The only way this equation can be solved is when:

$$0 = x+y-n \rightarrow x+y=n$$ //

2. $f(x) = \frac{x}{6} - \frac{x^2}{36}$; $0 \le x \le 6$

$= 0$; otherwise

(a). $F(x) = \int f(x)\,dx$

$F(x) = 0$; $x<0$

$= \frac{x^2}{12} - \frac{x^3}{108}$; $0 \le x \le 6$

$= 1$; $x>6$

(b). $E(x) = \int_{-\infty}^{\infty} x\left(\frac{x}{6} - \frac{x^2}{36}\right)dx$

$= \int_0^6 \left(\frac{x^2}{6} - \frac{x^3}{36}\right)dx$

$= 3$

$\mu_x = E(x) = 3$ //

$var(x) = E[(x-\mu)^2]$

$= \int_{-\infty}^{\infty} (x-3)^2\left(\frac{x}{6} - \frac{x^2}{36}\right)dx$

$= \int_0^6 (x-3)^2\left(\frac{x}{6} - \frac{x^2}{36}\right)dx$

$= 1.8$

$\sigma_x^2 = var(x) = 1.8$ //

(c). capacity = 3

P(overflow | X<4) = P(3<X<4) = F(4) − F(3) = 0.2407 //

(d). capacity = 4

P(replacement completed) = 0.7

P(overflow) = 0.3 × P(X>3) + 0.7 × P(X>4)

= 0.3(1−F(3)) + 0.7(1−F(4))

= 0.3315 //

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Date No. 3

3. $x_i = 1$: support $\quad i = 1, 2, \ldots, n$

$= 0$: don't support

support ratio, $p = P(X=1)$

(a). $E[\hat{p}] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$

$= \frac{1}{n}\sum_{i=1}^{n} E[x_i]$

$= \frac{1}{n} \times n \times E[x_i]$

$= p$ ~ shown!

$var[\hat{p}] = var\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$

$= \frac{1}{n^2}\sum_{i=1}^{n} var[x_i]$

$= \frac{1}{n^2}\sum_{i=1}^{n} \left(E[x_i^2] - E[x_i]^2\right)$ — $E[x_i^2] = E[x_i] = p$ because x_i can only be 0 or 1.

$= \frac{1}{n^2}\sum_{i=1}^{n} (p - p^2)$

$= \frac{p(1-p)}{n}$ ~ shown!

(b). 55 out of 100 participants → $p = 0.55$

$E[x] = E[\hat{p}] = p = 0.55$

$var[x] = var[\hat{p}] = \frac{p(1-p)}{n} = 0.002475$

(c). estimator for $var[x]$, $\hat{\sigma}_x^2 = \hat{p}(1-\hat{p}) = \bar{x}(1-\bar{x})$

To check whether the estimator is unbiased or not:

$\hat{\sigma}_x^2$ is unbiased estimator if $E[\hat{\sigma}_x^2] = var[x]$

$E[\hat{\sigma}_x^2] = E[\hat{p}(1-\hat{p})]$

$= E[\hat{p} - \hat{p}^2]$

$= E[\hat{p}] - E[\hat{p}^2]$

$= p - (var[p] + E[p]^2)$

$= p - \left(\frac{p-p^2}{n} + p^2\right)$

$= \frac{np - np^2 - (p-p^2)}{n}$

$= (n-1) \cdot \frac{p(1-p)}{n} \neq var[x]$ → so it is biased.

The unbiased one is : $\hat{\sigma}_x^2 = \frac{1}{n-1} \cdot \hat{p}(1-\hat{p})$

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Date   No. 4

3. (d). 95% CI for p. → $\bar{X} = 0.55$

$\alpha = 0.5$

By using Central Limit Theorem, as $n > 30$, we may assume p to be normally distributed.
So:

p-CI for p: $\bar{X} - z_{\alpha/2} \cdot \sigma/\sqrt{n} \leq p \leq \bar{X} + z_{\alpha/2} \cdot \sigma/\sqrt{n}$

$z_{\alpha/2} = z_{0.25} = 1.96$ (from table)

$\sigma = \sqrt{var[X]} = 0.049$

$n = 100 \rightarrow \sqrt{n} = 10$

$$\bar{X} - z_{\alpha/2} \cdot \sigma/\sqrt{n} \leq p \leq \bar{X} + z_{\alpha/2} \cdot \sigma/\sqrt{n}$$

$$0.55 - 1.96 \cdot \frac{0.049}{10} \leq p \leq 0.55 + 1.96 \cdot \frac{0.049}{10}$$

$$0.54 \leq p \leq 0.56$$ //

4. (a). 10 randomly selected components:

101 101.3 100.4 99.8 98.8

100.7 99.3 100.8 98.2 99.5 ~ assume normally distributed

$\bar{X} = 99.98$ $S = 0.9714$

p-CI 95% → $\alpha = 0.5$

o) mean: $H_0: \mu_0 = 100$ $H_1: \mu \neq 100$

$T_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ $t_{\alpha/2, n-1} \doteq t_{0.25, 9}$

$= -0.065$ $= 0.703$

Because -0.065 is not outside -0.703 and 0.703, we cannot reject $H_0: \mu_0 = 100$.

o) variance / standard deviation: $H_0: \sigma_0 \leq 2$ $H_1: \sigma > 2$

$V_0 = \frac{(n-1)s^2}{\sigma_0^2}$ $\chi^2_{\alpha/2, n-1} = \chi^2_{0.25, 9}$

$= 2.123$ $= 11.3$

As 2.123 is not larger than 11.3, we cannot reject $H_0: \sigma_0 \leq 2$.

THEREFORE, this batch is acceptable with 95% confidence level.

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Date No. 5

4. (b).(i) $Y = a_0 + a_1 X$

X	3	4	5	6	7
Y	20.1	24.6	30.3	36	39.4

$\bar{X} = 5 \qquad \bar{Y} = 30.08$

$S_{XY} = \sum_{k=1}^{n} X_k Y_k - n\bar{X}\bar{Y} = 802 - 5\times5\times30.08 = 50$

$S_{XX} = \sum_{k=1}^{n} X_k^2 - n\bar{X}^2 = 135 - 125 = 10$

$S_{YY} = \sum_{k=1}^{n} Y_k^2 - n\bar{Y}^2 = 4775.62 - 4524.032 = 251.6$

$\hat{a}_1 = \frac{S_{XY}}{S_{XX}} \qquad \hat{a}_0 = \bar{Y} - \hat{a}_1\bar{X}$

$= 5 \qquad = 5.08$

(ii) $a_0 = \hat{a}_0 \pm t_{(1-p)/2,\, n-2} \sqrt{\frac{SS_E}{n-2}\left(\frac{1}{n} + \frac{\bar{X}^2}{S_{XX}}\right)}$

90% - CI $\rightarrow \alpha = 0.1 \qquad t_{\alpha/2,\, n-2} = t_{0.05, 8} = 1.86$ (obtained from table)

$SS_E = S_{YY} - \hat{a}_1 S_{XY}$

$= 1.6 \qquad n = 10$

$a_0 = 5.08 \pm 1.86 \sqrt{\frac{1.6}{8}\left(\frac{1}{10} + \frac{25}{10}\right)}$

$= 5.08 \pm 1.34$

so: $3.74 < a_0 < 6.42$

- FIN -

Good luck for your exam !!