CV 4101 AY 2016-2017

1. a) 2 DOFs - u_4', v_4'

b) Member 1 $c=1$, $s=0$, $\frac{AE}{L} = 400\,000$

$$K_1' = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & 400\,000 & 0 \\ \times & \times & 0 & 0 \end{array}\right]$$

Member 2 $c=0$, $s=1$, $\frac{AE}{L} = 400\,000$

$$\cancel{K_2' = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & 0 & 0 \\ \times & \times & 0 & 400\,000 \end{array}\right]}$$

$$K_2' = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \boxed{\begin{matrix} 0 \\ -400\,000 \end{matrix}} & 0 & 0 \\ \times & & 0 & 400\,000 \end{array}\right]$$

$\searrow$ for part (c)

Member 3 $c=\frac{1}{\sqrt{2}}$, $s=\frac{1}{\sqrt{2}}$, $\frac{AE}{L} = 400\,000$

$$K_3' = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline \times & \times & 200\,000 & 200\,000 \\ \times & \times & 200\,000 & 200\,000 \end{array}\right]$$

Assembling

$$\begin{bmatrix} R_{1x}' \\ R_{1y}' \\ R_{2x}' \\ R_{2y}' \\ R_{3x}' \\ R_{3y}' \\ -100 \\ 200 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & \text{column 6} & & \\ & \downarrow & & \\ & 0 & 600\,000 & 200\,000 \\ & -400\,000 & 200\,000 & 600\,000 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4' \\ v_4' \end{bmatrix}$$

$$\begin{pmatrix} -100 \\ 200 \end{pmatrix} = \begin{pmatrix} 600\,000 & 200\,000 \\ 200\,000 & 600\,000 \end{pmatrix} \begin{pmatrix} u_4' \\ v_4' \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} u_4' \\ v_4' \end{pmatrix} = \begin{pmatrix} -3.125 \times 10^{-4} \\ 4.375 \times 10^{-4} \end{pmatrix} (m)$$

c) $v_4' = 2.1875 \times 10^{-4}\,m$

$$\begin{pmatrix} -100 \\ 200 \end{pmatrix} = \begin{pmatrix} 0 & 600\,000 & 200\,000 \\ -400\,000 & 200\,000 & 600\,000 \end{pmatrix} \begin{pmatrix} v_3' \\ u_4' \\ 2.1875\times10^{-4} \end{pmatrix}$$

$$\begin{pmatrix} -143.75 \\ 68.75 \end{pmatrix} = \begin{pmatrix} 0 & 600\,000 \\ -400\,000 & 200\,000 \end{pmatrix} \begin{pmatrix} v_3' \\ u_4' \end{pmatrix}$$

$v_3' = -2.917 \times 10^{-4}\,m$

magnitude of settlement = $2.917 \times 10^{-4}\,m$

2. a) 2 DOFs – θ_2', v_3'

b) $\frac{4EI_1}{L_1} = 1600$, $\frac{2EI_1}{L_1} = 800$, $\frac{3EI_2}{L_2} = 1200 = \frac{3EI_2}{L_2^2} = \frac{3EI_2}{L_2^3}$

$$K_1' = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & 800 \\ \times & \times & \times & \times \\ \times & \times & \times & 1600 \end{pmatrix}, \quad F_{01}' = \begin{pmatrix} \times \\ \frac{24\times1^2}{12} \\ \times \\ -\frac{24\times1^2}{12} \end{pmatrix} = \begin{pmatrix} \times \\ 2 \\ \times \\ -2 \end{pmatrix}$$

$$K_2' = \begin{pmatrix} \times & \times & \times & 0 \\ \times & 1200 & -1200 & 0 \\ \times & -1200 & 1200 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_{02}' = \begin{pmatrix} \times \\ \frac{3PL}{16} \\ \frac{5P}{16} \\ 0 \end{pmatrix} = \begin{pmatrix} \times \\ 4.5 \\ 7.5 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} R_{1y}' \\ M_{1\theta}' \\ R_{2y}' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} & \text{column 4} & \\ & \downarrow & \\ & 800 & \\ & & \text{column 5} \\ & & \downarrow \\ & 2800 & -1200 \\ & -1200 & 1200 \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \theta_2' \\ v_3' \\ \times \end{pmatrix} + \begin{pmatrix} \times \\ 2 \\ \times \\ 2.5 \\ 7.5 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2800 & -1200 \\ -1200 & 1200 \end{pmatrix} \begin{pmatrix} \theta_2' \\ v_3' \end{pmatrix} + \begin{pmatrix} 2.5 \\ 7.5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \theta_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} -6.25\times10^{-3} \\ -0.0125 \end{pmatrix} \begin{matrix} \text{rad} \\ m \end{matrix}$$

c) $M_{1\theta}' = 800\,\theta_2' + 2 = 0.01 \times \frac{200\times2}{1}$

$= 4$

$\theta_2' = 2.5 \times 10^{-3}$ rad

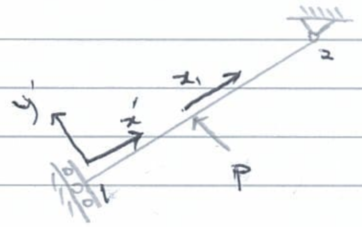
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2800 & -1200 \\ -1200 & 1200 \end{pmatrix} \begin{pmatrix} \theta_2' \\ v_3' \end{pmatrix} + \begin{pmatrix} \frac{3P}{16} - 2 \\ \frac{5P}{16} \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2800 & -1200 \\ -1200 & 1200 \end{pmatrix} \begin{pmatrix} 2.5\times10^{-3} \\ v_3' \end{pmatrix} + \begin{pmatrix} \frac{3P}{16} - 2 \\ \frac{5P}{16} \end{pmatrix}$$

$$-5 = -1200\,v_3' + \frac{3P}{16} \quad \cdots \text{①}$$

$$3 = 1200\,v_3' + \frac{5P}{16} \quad \cdots \text{②}$$

$$P = -4\ \text{kN}$$

3. a)



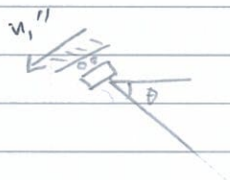
1 DOF – v_1'

$$K_1' = \left(\begin{array}{ccc|ccc} \times & \times & \times & \times & \times & 0 \\ \times & \frac{3EI}{L^3} & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & 0 \\ \hline \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{pmatrix} R_{1x'} \\ 0 \\ M_{1\theta'} \\ R_{2x'} \\ R_{2y'} \\ 0 \end{pmatrix} = \begin{array}{c} \text{column 2} \\ \downarrow \\ \frac{3EI}{L^3} \end{array} \begin{pmatrix} 0 \\ v_1' \\ 0 \\ 0 \\ 0 \\ \times \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{11P}{16} \\ \times \\ 0 \\ \times \\ 0 \end{pmatrix}$$

$$F_d' = \begin{pmatrix} 0 \\ -\frac{11P}{16} \\ \times \\ 0 \\ \times \\ 0 \end{pmatrix} \qquad 0 = \frac{3EI}{L^3} v_1' - \frac{11P}{16}$$

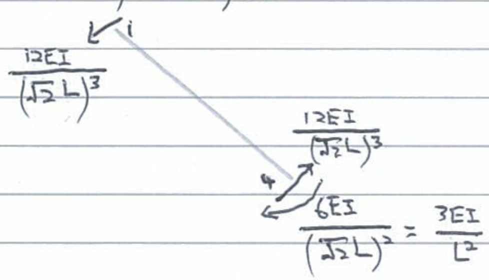
$$\therefore v_1' = \frac{11PL^3}{48EI}$$

b) 3 DOFs – u_1'', v_4', θ_4'

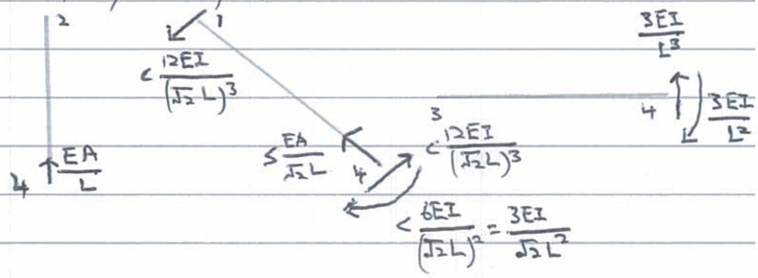


$s = \sin\theta$, $c = \cos\theta$

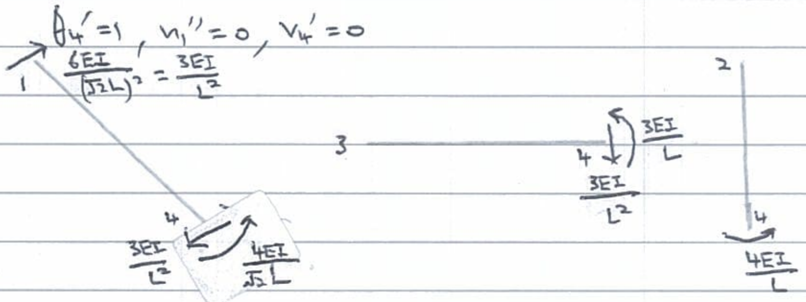
$u_1'' = 1$, $v_4' = 0$, $\theta_4' = 0$



$v_4' = 1$, $u_1'' = 0$, $\theta_4' = 0$

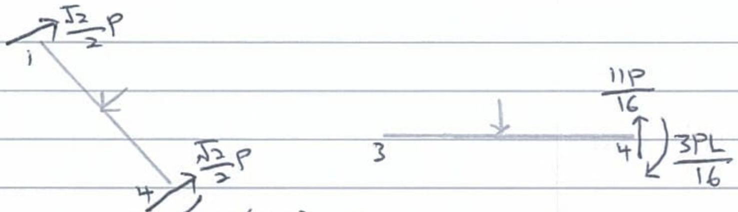


$\theta_4' = 1$, $u_1'' = 0$, $v_4' = 0$



$$
k_{ff}' = \begin{array}{c} \left(\begin{array}{c|c|c} 12\frac{EI}{(\sqrt{2}L)^3} & c\frac{12EI}{(\sqrt{2}L)^3} & -\frac{3EI}{L^2} \\ c\frac{12EI}{(\sqrt{2}L)^3} & \frac{EA}{L} + s^2\frac{EA}{\sqrt{2}L} + c^2\frac{12EI}{(\sqrt{2}L)^3} + \frac{3EI}{L^3} & -\frac{3EI}{\sqrt{2}L^2} - \frac{3EI}{L^2} \\ -\frac{3EI}{L^2} & -\frac{3EI}{\sqrt{2}L^2} - \frac{3EI}{L^2} & \left(7+\frac{4}{\sqrt{2}}\right)\frac{EI}{L} \end{array}\right) \end{array} \begin{array}{l} \leftarrow u_1'' \\ \leftarrow v_4' \\ \leftarrow \theta_4' \end{array}
$$

$$
= \begin{pmatrix} 4.2426\frac{EI}{L^3} & 3\frac{EI}{L^3} & -3\frac{EI}{L^2} \\ 3\frac{EI}{L^3} & 6.4749\frac{EI}{L^3} & -5.1213\frac{EI}{L^2} \\ -\frac{3EI}{L^2} & -5.1213\frac{EI}{L^2} & 9.8284\frac{EI}{L} \end{pmatrix}
$$



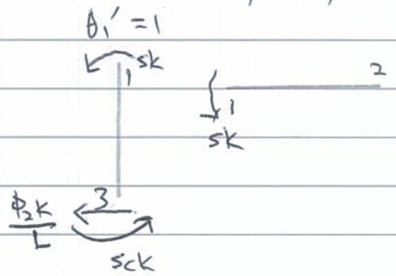
$$
\frac{\sqrt{2}P(\sqrt{2}L)}{8} = \frac{PL}{4}
$$

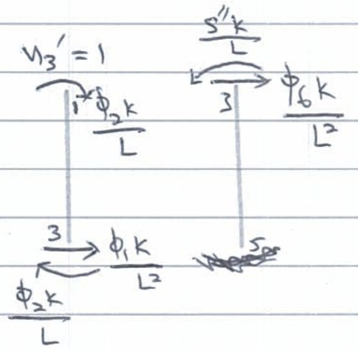
$$
F_{0k}' = \begin{pmatrix} -\frac{\sqrt{2}}{2}P \\ c\frac{\sqrt{2}}{2}P + \frac{11P}{16} \\ -\frac{PL}{4} - \frac{3PL}{16} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2}P \\ \frac{19}{16}P \\ -\frac{7}{16}PL \end{pmatrix}
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.2426\frac{EI}{L^3} & 3\frac{EI}{L^3} & -3\frac{EI}{L^2} \\ 3\frac{EI}{L^3} & 6.4749\frac{EI}{L^3} & -5.1213\frac{EI}{L^2} \\ -3\frac{EI}{L^2} & -5.1213\frac{EI}{L^2} & 9.8284\frac{EI}{L} \end{pmatrix} \begin{pmatrix} u_1'' \\ v_4' \\ \theta_4' \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{2}}{2}P \\ \frac{19}{16}P \\ -\frac{7}{16}PL \end{pmatrix}
$$

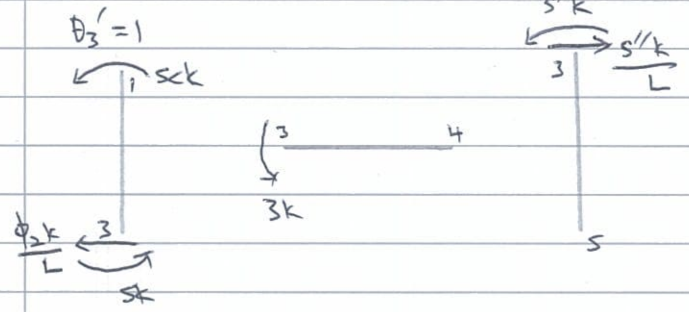
$$
\rightarrow \begin{pmatrix} u_1'' \\ v_4' \\ \theta_4' \end{pmatrix} = \begin{pmatrix} 0.4320 \\ -0.4151 \\ -0.03996/L \end{pmatrix} \frac{PL^3}{EI}
$$

4. a) 3 DOFs – θ_1', u_3', θ_3' ($u_3' = u_4'$)





$k = \frac{EI}{L}$



$$
k_{ff}' = \begin{pmatrix} 2sk & -\frac{\phi_2 k}{L} & sck \\ -\frac{\phi_2 k}{L} & \frac{\phi_1 k}{L^2} + \frac{\phi_6 k}{L^2} & \frac{s''k}{L} - \frac{\phi_2 k}{L} \\ sck & \frac{s''k}{L} - \frac{\phi_2 k}{L} & (3+s+s'')k \end{pmatrix} \begin{array}{l} \leftarrow \theta_1' \\ \leftarrow u_3' \\ \leftarrow \theta_3' \end{array}
$$

$P_{12} = P_{13} = P_{35} = P/\sqrt{2}$

$P_{E,12} = P_{E,13} = P_{E,35} = \frac{\pi^2 EI}{L^2}$, so $\rho_{12} = \rho_{13} = \rho_{35} = \rho$

$$|k_{ff}'| = 2sk\begin{vmatrix} \frac{\phi_1 k}{L^2}+\frac{\phi_6 k}{L^2} & \frac{s''k}{L}-\frac{\phi_2 k}{L} \\ \frac{s''k}{L}-\frac{\phi_2 k}{L} & (3+s+s'')k \end{vmatrix} - \left(-\frac{\phi_2 k}{L}\right)\begin{vmatrix} -\frac{\phi_2 k}{L} & \frac{s''k}{L}-\frac{\phi_2 k}{L} \\ sck & (3+s+s'')k \end{vmatrix} + sck\begin{vmatrix} -\frac{\phi_2 k}{L} & \frac{\phi_1 k}{L^2}+\frac{\phi_6 k}{L^2} \\ sck & \frac{s''k}{L}-\frac{\phi_2 k}{L} \end{vmatrix} = 0$$

$$\frac{2sk^3}{\cancel{L^2}}\left[(\phi_1+\phi_6)(3+s+s'') - (s''-\phi_2)^2\right] + \frac{\phi_2 k^3}{\cancel{L^2}}\left[-\phi_2(3+s+s'') - sc(s''-\phi_2)\right]$$

$$+ sc\frac{k^3}{\cancel{L^2}}\left[-\phi_2(s''-\phi_2) - sc(\phi_1+\phi_6)\right] = 0$$

$$\therefore (3+s+s'')(2s\phi_1 + 2s\phi_6 - \phi_2^2) - 2sc\phi_2(s''-\phi_2) - (sc)^2(\phi_1+\phi_6) - 2s(s''-\phi_2)^2 = 0$$

$\rho = 0.40$, $s = 3.4439$, $c = 0.6242$, $s'' = 2.1021$, $f(s,c) = 25.1708$

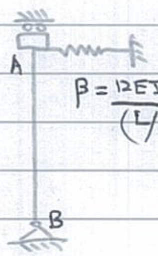
$\rho = 0.42$, $s = 3.4144$, $c = 0.6321$, $s'' = 2.0501$, $f(s,c) = -2.1926$

$$\rho_{cr} = 0.40 + \frac{0-25.1708}{-2.1926-25.1708}(0.42-0.40)$$

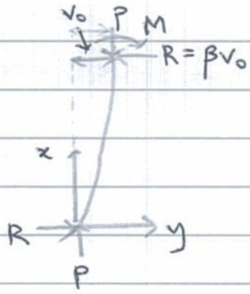
$$= 0.4184$$

$$\therefore P_{cr} = 0.4184 P_E \times \sqrt{2} = 0.5917\frac{\pi^2 EI}{L^2}$$

4. b)



$$\beta = \frac{12EI}{(L/2)^3} = \frac{96EI}{L^3}$$



$$EIv'' = -Pv + \beta v_0 x$$

$$v'' + \omega^2 v = \frac{\beta v_0}{EI}x, \quad \omega = \sqrt{\frac{P}{EI}}$$

$$v = A\sin\omega x + B\cos\omega x + \frac{\beta v_0}{P}x$$

$$\rightarrow v'(x) = A\omega\cos\omega x - B\omega\sin\omega x + \frac{\beta v_0}{P}$$

Apply boundary condition,

$$v(0) = 0, \quad B = 0$$

$$v'(L) = 0, \quad A\omega\cos\omega L + \frac{\beta v_0}{P} = 0$$

$$A = -\frac{\beta v_0}{P\omega\cos\omega L}$$

$$v(L) = v_0$$

$$-\frac{\beta \cancel{v_0}}{P\omega\cos\omega L}\sin\omega L + \frac{\beta \cancel{v_0}}{P}L = \cancel{v_0}$$

$$-\frac{96}{(\omega L)^3}\tan\omega L + \frac{96}{(\omega L)^2} = 1$$

$$\varphi^3 + 96(\tan\varphi - \varphi) = 0, \quad \varphi = \omega L$$

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