NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2011-2012

CV2002 – COMPUTATIONAL METHODS

April – May 2012 **Time Allowed: 2½ hours**

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL FOUR (4)** questions.
3. All questions carry equal marks.
4. This is an Open Book Examination restricted to **ONE (1)** sheet of A4 size paper containing any reference materials.

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1. (a) Given $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & 12 \\ 2 & 0 & p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ q \end{bmatrix}$

(i) Find the value of p and q for unique solution, multiple solution and no solution.

(ii) Find the multiple solution.

(9 marks)

(b) Given $\mathbf{B} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $|\mathbf{B}| = 3$, and $|\mathbf{C}| = 9$, where **C** is a 3x3 matrix.

Find the values of the determinants

(i) $\begin{vmatrix} b_1 + c_1 & a_1 + c_1 & 2b_1 \\ b_2 + c_2 & a_2 + c_2 & 2b_2 \\ b_3 + c_3 & a_3 + c_3 & 2b_3 \end{vmatrix}$

(ii) $\begin{vmatrix} a_1 & -2a_2 & a_3 \\ b_1 & -2b_2 & b_3 \\ -4c_1 & 8c_2 & -4c_3 \end{vmatrix}$

(iii) $|(2\mathbf{B})^{-1}|$

(iv) $|4\mathbf{B}^T|$

(v) $|3\mathbf{C}^{-1}\mathbf{B}^3\mathbf{C}^2\mathbf{B}^{-1}(\mathbf{C}^{-1})^T|$

(16 marks)

2. (a) Given $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$, find

(i) $\mathbf{A}^{-1}$

(ii) $\|7\mathbf{A}\|$

(iii) Rank($\mathbf{A}^{30} - 5\mathbf{A}^{29}$).

(10 marks)

(b) Given $\mathbf{D} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 6 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ and that $\begin{bmatrix} -3 \\ y \\ 4 \end{bmatrix}$ is an eigenvector of **D** corresponding to λ_1, where y is an unknown.

(i) Find the values of y and λ_1.

(ii) Given $\lambda_2 = 2$, find the value of λ_3.

(iii) Find the eigenvector corresponding to λ_2.

(iv) Find any one of the eigenvalues of $(\mathbf{D} + 2\mathbf{I})^{-1}$.

(15 marks)

3. (a) To test the brake of a car, the position of the car on a track is timed during an experiment:

t (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
x (m)	0.00	18.75	35.00	48.75	60.00	68.75	75.00

where t is the time after the brake is applied fully, and x is the distance travelled by the car after the brake is applied fully. Using appropriate numerical methods, estimate the velocities of the car at t = 1.0 s and at x = 50 m. Estimate the time when the car would stop completely.

(15 marks)

(b) A study requires the estimate of the number of people entering an MRT station, from 6 A.M. to 12 noon. A surveyor is assigned to visit the station at various times during a morning and count the number of people entering the station in a minute. The collected data is summarised in the table below. Using appropriate numerical methods, give the best estimate of the total number of people entering the station, from 6 A.M. to 12 noon.

Number of people entering the station per minute observed at various times in a morning.

Time	**Rate (No. of people per minute)**
06:00	120
07:00	240
07:30	300
08:00	420
08:30	560
09:30	320
10:30	200
11:00	180
12:00	220

(10 marks)

4. The differential equation governing the free vibration of an undamped single-degree-of-freedom system can be expressed as

$$\frac{d^2u}{dt^2} + \omega_n^2 u = 0$$

where u = the displacement of the mass
t = time
ω_n = the natural frequency of the system

The analytical solution of the differential equation under a certain initial condition is given as

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

where $u(0)$ = initial displacement (displacement at $t = 0$)

$\dot{u}(0) = \frac{du}{dt}(0)$ = initial velocity (velocity at $t = 0$)

The system has the following parameters:

ω_n = 2π rad/s
$u(0)$ = 0.2 m
$\dot{u}(0)$ = 4 m/s

(a) Using the analytical solution, calculate the displacement of the mass u at $t = 0.1$ s. Express the value in m and correct to three significant figures.

(3 marks)

(b) Taking a step size $\Delta t = 0.1$ s, calculate the displacement of the mass u at $t = 0.1$ s by using:

(i) the standard Euler's method
(ii) the midpoint method
(iii) the Heun's method
(iv) the fourth-order Range-Kutta method.

Express the values in m and correct to three significant figures. Compute the true fractional error for each method.

(22 marks)

END OF PAPER

U1040024L
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CV2002 - Computational Methods (2011-2012) S2

1. a) i) $[A|b] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 6 & 12 & 6 \\ 2 & 0 & P & q \end{bmatrix}$

$\xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 6 & 12 & 6 \\ 0 & 0 & p+2 & q-2 \end{bmatrix}$

For unique solution,

$p+2 \neq 0$ $\quad$ $q = \{-\infty, \infty\}$

$p \neq 2$

For multiple solution,

$p+2=0$, $q-2=0$

$p=-2$ $\quad$ $q=2$

For no solution,

$p+2=0$ $\quad$ $q-2 \neq 0$

$p=-2$ $\quad$ $q \neq 2$

ii) when $p=-2$, $q=2$, have multiple solution.

set $x_3 = \alpha$

$x_1 - x_3 = 1$

$x_1 = 1 + \alpha$

$6x_2 + 12x_3 = 6$

$x_2 + 2x_3 = 1$

$x_2 = 1 - 2\alpha$

$\therefore x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

$\begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix} = 0$

b) i) $\begin{vmatrix} b_1+c_1 & a_1+c_1 & 2b_1 \\ b_2+c_2 & a_2+c_2 & 2b_2 \\ b_3+c_3 & a_3+c_3 & 2b_3 \end{vmatrix} = \begin{vmatrix} b_1 & a_1+c_1 & 2b_1 \\ b_2 & a_2+c_2 & 2b_2 \\ b_3 & a_3+c_3 & 2b_3 \end{vmatrix} + \begin{vmatrix} c_1 & a_1+c_1 & 2b_1 \\ c_2 & a_2+c_2 & 2b_2 \\ c_3 & a_3+c_3 & 2b_3 \end{vmatrix}$

$= 0 + \begin{vmatrix} c_1 & c_1 & 2b_1 \\ c_2 & c_2 & 2b_2 \\ c_3 & c_3 & 2b_3 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 & 2b_1 \\ c_2 & a_2 & 2b_2 \\ c_3 & a_3 & 2b_3 \end{vmatrix}$

$= 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= 2|B|$

$= 2(3)$

$= 6$

ii) $\begin{vmatrix} a_1 & -2a_2 & a_3 \\ b_1 & -2b_2 & b_3 \\ -4c_1 & 8c_2 & -4c_3 \end{vmatrix} = -4 \begin{vmatrix} a_1 & -2a_2 & a_3 \\ b_1 & -2b_2 & b_3 \\ c_1 & -2c_2 & c_3 \end{vmatrix}$

$= -4\times(-2) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= -4\times(-2)\times|B^T|$

$= 8\times|B|$

$= 8\times 3$

$= 24$

iii) $|(2B)^{-1}| = \frac{1}{|2B|}$

$= \frac{1}{2^3}\times\frac{1}{|B|}$

$= \frac{1}{8\times 3}$

$= \frac{1}{24}$

iv) $|4B^T| = 4^3|B^T|$

$= 4^3|B|$

$= 4^3(3)$

$= 192$

v) $|3C^{-1}B^3C^2B^{-1}(C^{-1})^T| = 3^3\times\cancel{\frac{1}{|C|}}\times|B|^{\cancel{3}\,2}\times\cancel{|C|^2}\times\cancel{\frac{1}{|B|}}\times\cancel{|C^{-1}|}$

$= 27\times|B|^2$

$= 27(3)^2$

$= 243$

2. a) i) $|A| = 2 \neq 0$,

$\therefore A^{-1}$ exist.

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \end{bmatrix}^T$

$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & -2 & 0 \end{bmatrix}^T$

$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 5 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

ii) $\|7A\| = |7|\|A\|$

$= 7\sqrt{2^2+1^2+(-1)^2+5^2}$

$= 38.97$

iii) $|A^{30} - 5A^{29}| = |A^{29}(A-5I)|$

$= |A^{29}||A-5I|$

$$= |A|^{29}\begin{vmatrix} 2-5 & -5 & -5 \\ -5 & -5 & -1-5 \\ -5 & 1-5 & 5-5 \end{vmatrix}$$

$$= |A|^{29}\begin{vmatrix} -3 & -5 & -5 \\ -5 & -5 & -6 \\ -5 & -4 & 0 \end{vmatrix}$$

$= |A|^{29}[-150 - 100 - (-125 - 72)]$

$= (-53)(2)^{29} \neq 0$

$\therefore |A^{30} - 5A^{29}| \neq 0$

$\therefore \text{Rank}(A^{30} - 5A^{29}) = n$

$= 3$

b) $Dx = \lambda_1 x$

$$\begin{bmatrix} -1 & 3 & 0 \\ -2 & 6 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ y \\ 4 \end{bmatrix} = \lambda_1\begin{bmatrix} -3 \\ y \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3+3y \\ 6+6y \\ 6-y+8 \end{bmatrix} = \begin{bmatrix} -3\lambda_1 \\ \lambda_1 y \\ 4\lambda_1 \end{bmatrix}$$

$3+3y = -3\lambda_1$

$1+y = -\lambda_1$

$y = -\lambda_1 - 1$ — ①

$6-y+8 = 4\lambda_1$ put ① into

$6+\lambda_1+1+8 = 4\lambda_1$

$3\lambda_1 = 15$

$\lambda_1 = 5$ into ①

$y = -5-1$

$= -6$

ii) $\sum a_{ii} = \sum \lambda_i$

$-1+6+2 = \lambda_1+\lambda_2+\lambda_3$

$7 = 2+5+\lambda_3$

$\lambda_3 = 0$

iii) $Dx = \lambda_2 x$

$Dx - \lambda_2 x = 0$.

$(D - \lambda_2 I)x = 0 \Rightarrow (D - 2I)x = 0$

set eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} -1-2 & 3 & 0 \\ -2 & 6-2 & 0 \\ -2 & -1 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 3 & 0 \\ -2 & 4 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3x_1 + 3x_2 \\ -2x_1 + 4x_2 \\ -2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 = 0$

$x_1 = x_2$ — ①

$-2x_1 + 4x_2 = 0$

$-2x_2 + 4x_2 = 0$

$2x_2 = 0$

$x_2 = 0$ into ①

$x_1 = x_2$

$= 0$

set $x_3 = \alpha$

$\therefore x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \alpha$

iv) $Dx = \lambda x$

$Dx + 2Ix = \lambda x + 2Ix$

$(D+2I)x = (\lambda+2)x$

$Ix = (D+2I)^{-1}(\lambda+2)x$

$\frac{1}{\lambda+2} x = (D+2I)^{-1} x$

$(D+2I)^{-1} = \frac{1}{\lambda+2} x$

$\therefore$ any one of the eigenvalues of $(D+2I)^{-1}$

$= \frac{1}{2+2}$

$= \frac{1}{4}$

3.a) $x = f(t)$

$f(1) = 35.00$

$f(0.5) = 18.75$

$f(1.5) = 48.75$

velocity of the car at $t = 1.0s$

$= \frac{d}{dt} x(1.0)$

$= f'(1.0)$

$= \frac{f'(t_{i+1}) - f(t_{i-1})}{2h}$

$= \frac{48.75 - 18.75}{2 \times 0.5}$

$= 30 \ ms^{-1}$

t	f(t)	first	second	third
1.0	35.00	27.5	-5	0
1.5	48.75	22.5	-5	
2.0	60.00	17.5		
2.5	68.75			

$f(t) = 35 + 27.5(t-1) - 5(\ -1)(t-1.5)$

$= 35 + 27.5t - 27.5 - 5(t^2 - 2.5t + 1.5)$

$= 35 + 27.5t - 27.5 - 5t^2 + 12.5t - 7.5$

$= -5t^2 + 40t$

when $f(t) = 50$

$50 = -5t^2 + 40t$

$t^2 - 8t + 10 = 0$

$t = \frac{8 \pm \sqrt{64 - 40}}{2}$

$= 6.45 \ s$ (no suitable) or $1.55s$ (suitable)

$f'(t) = -10t + 40$

$f'(1.55) = 24.5 \ ms^{-1}$

the velocity of the car at $x = 50m$

$= 24.5 \ ms^{-1}$

$f'(t) = 0$

$0 = -10t + 40$

$t = 4s$

The car would stop completely at $t = 4$ s.

3.(b)

Time	Rate	No. of people
6.00	120	$(60)\times\frac{120+240}{2} = 10800$
7.00	240	$(90)\times\frac{240+3(300)+3(420)+560}{8} = 33300$
7.30	300	
8.00	420	
8.30	560	$(120)\times\frac{560+4(320)+200}{6} = 40800$
9.30	320	
10.30	200	$(30)\times\frac{200+180}{2} = 5700$
11:00	180	$(60)\times\frac{180+220}{2} = 12000$
12:00	220	
	Total	102600

The total number of people entering the station is 102600 people.

4. a) $u(t) = u(0)\cos(\omega_n t) + \frac{\dot{u}(0)}{\omega_n}\sin(\omega_n t)$

$= 0.2\cos(2\pi t) + \frac{4}{2\pi}\sin(\omega_n t)$

when $t = 0.1$

$u(0.1) = 0.2\cos(2\pi\times 0.1) + \frac{4}{2\pi}\sin(2\pi\times 0.1)$

$= 0.536$

b) i) set $z = \frac{dy}{dt}$,

$\frac{dz}{dt} = -\omega_n^2 u$

t	u_i	$\frac{dz}{dt}$	$(z/\frac{dy}{dt})_{i+1}$	u_{i+1}	ε_t
0	0.2	-7.896	3.210	0.521	2.8%

when $t=0$, $u=0.2$

$\frac{dz}{dt} = -(2\pi)^2(0.2)$

$= -7.896$

$z_{i+1} = z_0 + \frac{dz}{dt}(\Delta t)$ ← $= \frac{dy}{dt}_0 = \dot{u}(0)$

$= 4 + (-7.896)(0.1)$

$= 3.210$

$u_{i+1} = u_0 + \frac{dy}{dt}(\Delta t)$

$= 0.2 + (3.210)(0.1)$

$= 0.521$

$\varepsilon_t = \frac{|0.536 - 0.521|}{0.536}$

$= 2.8\%$

ii)

t	u	$\frac{dz}{dt}$	$z_{i+\frac{1}{2}}/\frac{dy}{dt}_{i+\frac{1}{2}}$	u_{i+1}	ε
0	0.2	-7.896	3.605	0.561	-4.66%

$z_{i+\frac{1}{2}} = z_0 + \frac{dz}{dt}(\frac{1}{2}\Delta t)$

$= 4 - 7.896(\frac{1}{2})(0.1)$

$= 3.605$

$u_{i+1} = u_0 + \frac{dy}{dt}(\Delta t)$

$= 0.2 + (3.605)(0.1)$

$= 0.5605$

$\varepsilon = \frac{0.536 - 0.561}{0.536}$

$= -4.66\%$

iii)

t	u	$\frac{dz}{dt}$	z'_{i+1}	z_i	Ave z	u_{i+1}	ϵ
0	0.2	-7.896	3.21	4	3.605	0.561	-4.66%

$$\text{Ave } z = \frac{z'_{i+1}+z_i}{2} = \frac{4+3.21}{2} = 3.605$$

iv)

t	u_i	z_i	$\frac{dz}{dt}$	$z'_{i+\frac{1}{2}}$	$u'_{i+\frac{1}{2}}$	$\left(\frac{dz}{dt}\right)'_{i+\frac{1}{2}}$	$z''_{i+\frac{1}{2}}$	$u''_{i+\frac{1}{2}}$	$\left(\frac{dz}{dt}\right)''_{i+\frac{1}{2}}$	z'_{i+1}	Ave z	u_{i+1}	ϵ
0	0.2	4	-7.846	3.605	0.380	-15	3.25	0.363	-14.331	2.567	3.380	0.538	-0.37%

$$z'_{i+\frac{1}{2}} = z_i + \frac{dz}{dt}\left(\frac{\Delta t}{2}\right) = 4 - 7.896(0.05) = 3.605$$

$$u'_{i+\frac{1}{2}} = u_i + \left(\frac{du}{dt}\right)'_{i+\frac{1}{2}}\left(\frac{\Delta t}{2}\right) = 0.2 + (3.605)(0.05) = 0.380$$

$$\left(\frac{dz}{dt}\right)'_{i+\frac{1}{2}} = -\omega_n^2 u'_{i+\frac{1}{2}} = -(2\pi)^2(0.38) = -15$$

$$z''_{i+\frac{1}{2}} = z_i + \left(\frac{dz}{dt}\right)'_{i+\frac{1}{2}}\left(\frac{\Delta t}{2}\right) = 4 - 15(0.05) = 3.25$$

$$u''_{i+\frac{1}{2}} = u_i + \left(\frac{du}{dt}\right)''_{i+\frac{1}{2}}\left(\frac{\Delta t}{2}\right) = 0.2 + 3.25(0.05) = 0.363$$

$$\left(\frac{dz}{dt}\right)''_{i+\frac{1}{2}} = -\omega_n^2 u''_{i+\frac{1}{2}} = (2\pi)^2(0.363) = -14.331$$

$$z'_{i+1} = z_i + \left(\frac{dz}{dt}\right)''_{i+\frac{1}{2}}(\Delta t) = 4 - 14.331(0.1) = 2.567$$

$$\text{Ave } z = \frac{z_i + 2z'_{i+\frac{1}{2}} + 2z''_{i+\frac{1}{2}} + z'_{i+1}}{6} = \frac{4 + 2\times 3.605 + 2\times 3.25 + 2.567}{6} = 3.380$$

$$u_{i+1} = u_i + \text{Ave } z\,\Delta t = 0.2 + 3.380(0.1) = 0.538$$

$$\epsilon = \frac{0.536-0.538}{0.536} = -0.37\%$$