NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2009-2010

CV2002 – COMPUTATIONAL METHODS

April – May 2010 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FOUR (4)** pages.
2. Answer **ALL FOUR (4)** questions.
3. All questions carry equal marks.
4. This is an open book examination restricted to **ONE (1)** sheet of A4 size paper containing any reference materials.

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1. (a) If $|A| = -1$, $|B| = 2$ and $|C| = 3$, calculate

(i) $|A^3 B C^T B^{-1}|$ and

(ii) $|B^2 C^{-1} A B^{-1} C^T|$.

(6 marks)

(b) If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -2$ calculate $\begin{vmatrix} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{vmatrix}$.

(6 marks)

(c) If $A = \begin{bmatrix} a & b & c & d \\ a & b & c & -d \\ a & b & -c & -d \\ a & -b & -c & -d \end{bmatrix}$ find $|A|$.

(7 marks)

(d) If $B = \begin{bmatrix} a & a+b & a+2b \\ a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \end{bmatrix}$ show that $\text{rank}(B) < 3$.

(6 marks)

2. (a) Construct an augmented matrix from the following equation. Using elementary row operations reduce the matrix to echelon (triangular) form.

$$\begin{pmatrix} 1 & -2 & 2 \\ 4 & -7 & a \\ 3 & a & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Hence, or otherwise, determine the values of "**a**" so that the equation has

(i) no solution,
(ii) multiple solutions,
(iii) a unique solution.

Solve the equation for part (ii) above.

(11 marks)

(b) If $\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

(i) find the eigenvalues of $\mathbf{A}$ and the associated eigenvectors;
(ii) find the eigenvalues of $\mathbf{A}^{10}$ and the associated eigenvectors;
(iii) find $\|\mathbf{A}\|$.

(10 marks)

(c) Prove that a square matrix $\mathbf{A}$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$.

(4 marks)

3. (a) The elevations (in m) of a field are measured at various grid points and the results are summarized in Table Q3.

Use quadratic interpolation to estimate the elevation at $x = 3$, $y = 3$.

(15 marks)

Table Q3

	$x = 0$	$x = 2$	$x = 4$
$y = 0$	10	18	42
$y = 2$	12	20	44
$y = 4$	6	14	38

(b) The plot of function $f(x)$ below is given in Figure Q3. By observing the figure, the maximum value of $f(x)$ occurs when x is between 3 and 4. Using an appropriate numerical method, compute the value of x that gives maximum value of $f(x)$. Stop the iterations when the value of x is accurate up to three significant figures.

$$f(x) = 5 + 40x - 6x^2 + 0.1x^3 - 0.01x^4$$

(10 marks)

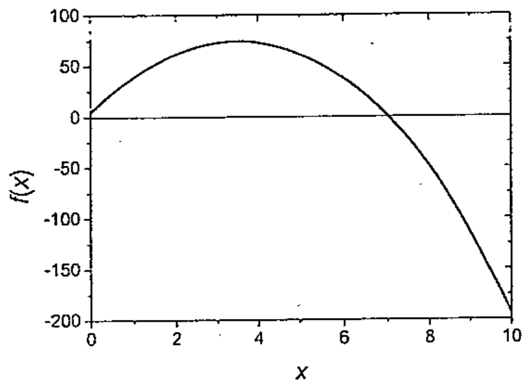

Figure Q3

4. The differential equation of the deflection curve for a cantilever beam subjected to a uniformly distributed load q (Figure Q4) is given as

$$EI\frac{d^2v}{dx^2} = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}.$$

The modulus of elasticity of the material E = 200 GPa (200 × 10^9 N/m^2), the second moment of area of the cross section I = 300 × 10^6 mm^4, the length of the beam L = 4 m, and the load q = 20 kN/m. Note that the boundary conditions dictate that both the deflection and the slope of deflection at the fixed end are equal to zero.

(a) Knowing that the analytical solution is $v = -\frac{qx^2}{24EI}\left(6L^2 - 4Lx + x^2\right)$, compute the deflection v at points B and C. Express the values in mm and correct up to three significant figures.

(5 marks)

Δx

(b) Using a step size h = 2 m, solve the deflection of the beam using:

(i) the standard Euler's method,
(ii) the midpoint method.

Express the values in mm and correct up to three significant figures. Compute the true fractional error for each method.

(20 marks)

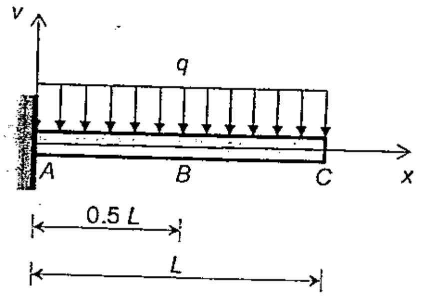

Figure Q4

END OF PAPER

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1. (a) (i) $|A^3BC^TB^{-1}| = |A|^3 \cdot |B| \cdot |C| \cdot \frac{1}{|B|} = (-1)^3 \times 2 \times 3 \times \frac{1}{2} = -3$

(ii) $|B^2C^{-1}AB^{-1}C^T| = |B|^2 \cdot \frac{1}{|C|} \cdot |A| \cdot \frac{1}{|B|} \cdot |C| = |B| \cdot |A| = -2$

(b) $\begin{vmatrix} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{vmatrix} = a_{31}|M_{31}| - a_{32}|M_{32}| + a_{33}|M_{33}|$

$= (a+1)\begin{vmatrix} 0 & 4d \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 2b & 4d \\ 1 & -2 \end{vmatrix} + 2(c-1)\begin{vmatrix} 2b & 0 \\ 1 & 2 \end{vmatrix}$

$= (a+1)(-8d) - 2(-4b-4d) + 2(c-1)(4b)$

$= -8ad - 8d + 8b + 8d + 8bc - 8b$

$= 8(bc-ad)$

$\because ad-bc=-2 \quad \therefore bc-ad=2 \quad \therefore \begin{vmatrix} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{vmatrix} = 16$

(c) row1 → Row1

row1 + row2 → Row 2

row1 + row3 → Row 3

row1 + row4 → Row 4

$|A| = \begin{vmatrix} a & b & c & d \\ 2a & 2b & 2c & 0 \\ 2a & 2b & 0 & 0 \\ 2a & 0 & 0 & 0 \end{vmatrix} = 8abcd$

(d) row1 → Row1

(-1) row1 + row2 → Row2

(-1) row1 + row3 → Row3

$|B| = \begin{vmatrix} a & a+b & a+2b \\ b & b & b \\ 2b & 2b & 2b \end{vmatrix}$

(-2) Row2 + Row3 → Row3 $\quad |B| = \begin{vmatrix} a & a+b & a+2b \\ b & b & b \\ 0 & 0 & 0 \end{vmatrix} = 0.$

$\therefore \text{rank}(B) < 3$

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2 (a) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 4 & -7 & a & 5 \\ 3 & a & -7 & 3 \end{array}\right] \xrightarrow[-3row1+row3 \to row3]{4row1 - row2 \to row2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & -1 & 8-a & 3 \\ 0 & 6+a & -13 & -3 \end{array}\right] \xrightarrow{row2\cdot(6+a)+row3 \to row3}$

$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & -1 & 8-a & 3 \\ 0 & 0 & -a^2+2a+35 & 3a+15 \end{array}\right]$

(i) $\begin{cases} -a^2+2a+35=0 \\ 3a+15 \neq 0 \end{cases} \Rightarrow \begin{cases} a=7 \text{ or } a=-5 \\ a \neq -5 \end{cases} \Rightarrow \{a=7$

(ii) $\begin{cases} -a^2+2a+35=0 \\ 3a+15=0 \end{cases} \Rightarrow \begin{cases} a=7 \text{ or } a=-5 \\ a=-5 \end{cases} \Rightarrow \{a=-5$

(iii) $-a^2+2a+35 \neq 0 \Rightarrow a \neq 7$ and $a \neq -5$

(b)(i) $|A-\lambda I|=0$ $\quad \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda)(3-\lambda)+2(2-\lambda)=0$

$\lambda_1=2 \quad \lambda_2=1$

when $\lambda_1=2$ $\quad A-\lambda I = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad x_1 = \begin{bmatrix} \alpha \\ \beta \\ -\alpha \end{bmatrix}$

when $\lambda_2=1$ $\quad A-\lambda I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

(ii) $A^2 = \begin{bmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{bmatrix} \quad A^3 = A \cdot A^2 = \begin{bmatrix} -6 & 0 & -14 \\ 7 & 8 & 7 \\ 7 & 0 & 15 \end{bmatrix}$

$A^5 = A^2 \cdot A^3 = \begin{bmatrix} -30 & 0 & -62 \\ 31 & 32 & 31 \\ 31 & 0 & 63 \end{bmatrix} \quad A^{10} = A^5 \cdot A^5 = \begin{bmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{bmatrix}$

$|A^{10}-\lambda I|=0 \quad \begin{vmatrix} -1022-\lambda & 0 & -2046 \\ 1023 & 1024-\lambda & 1023 \\ 1023 & 0 & 2047-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1=1 \quad \lambda_2=1024$

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when $\lambda_1 = 1$ $\quad A^{10} - \lambda I = \begin{bmatrix} -1023 & 0 & -2046 \\ 1023 & 1023 & 1023 \\ 1023 & 0 & 2046 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \alpha$

when $\lambda_2 = 1024$ $\quad A^{10} - \lambda I = \begin{bmatrix} -2046 & 0 & -2046 \\ 1023 & 0 & 1023 \\ 1023 & 0 & 1023 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} \alpha \\ \beta \\ -\alpha \end{bmatrix}$

(iii) $\|A\| = \sqrt{(-2)^2+1^2+2^2+1^2+1^2+3^2} = 2\sqrt{5}$

(c) (i) A is invertible $\Rightarrow |A| \neq 0$

For a eigenvalue λ, $|A - \lambda I| = 0 \Rightarrow \lambda$ cannot be zero.

$\therefore \lambda = 0$ is not an eigenvalue of A.

(ii) If $\lambda = 0$ is not an eigenvalue of A $\Rightarrow \lambda \neq 0$

$0 = |A - \lambda I| \neq |A|$

$\therefore |A| \neq 0$

$\therefore$ A is invertible

3 (a) when $y = 0$

x_i	$f(x_i)$	First	Second
0	10	$\frac{18-10}{2-0} = 4$	$\frac{12-4}{4-0} = 2$
2	18	$\frac{42-18}{4-2} = 12$	
4	42		

$f(x) = 10 + 4(x-0) + 2(x-0)(x-2)$

$f(3) = 28$

when $y = 2$

x_i	$f(x_i)$	First	Second
0	12	$\frac{12-20}{0-2} = 4$	$\frac{12-4}{4-0} = 2$
2	20	$\frac{44-20}{4-2} = 12$	
4	44		

$f(x) = 12 + 4x + 2x(x-2)$

$f(3) = 30$

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when $y=4$

x_i	$f(x_i)$	First	Second
0	6	$\frac{14-6}{2-0}=4$	$\frac{12-4}{4-0}=2$
2	14	$\frac{38-14}{4-2}=12$	
4	38		

$f(x)=6+4x+2x(x-2)$

$f(3)=24$

	$x=3$
$y=0$	28
$y=2$	30
$y=4$	24

y_i	$f(y_i)$	First	Second
0	28	$\frac{30-28}{2}=1$	$\frac{-3-1}{4-0}=-1$
2	30	$\frac{24-30}{4-2}=-3$	
4	24		

$f(y)=28+y-y(y-2)$

$f(3)=28$

$\therefore$ The elevation is 28 at $x=3$, $y=3$

(b) Let $g(x)=f'(x)=40-12x+0.3x^2-0.04x^3=0$ $\quad g(3)=5.62 \quad g(4)=-5.76$

$x_r=4-\frac{g(4)(3-4)}{g(3)-g(4)}=3.494 \quad g(3.494)=0.02822$

$x_r=4-\frac{g(4)(3.494-4)}{g(3.494)-g(4)}=3.496 \quad g(3.496)=5.478\times10^{-3}$

$x_r=4-\frac{g(4)(3.496-4)}{g(3.496)-g(4)}=3.496$

$\therefore x_r=3.50$

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4 (a) At point B,

$$V(B) = -\frac{q\frac{L^2}{4}}{24EI}\left(6L^2 - 4L\cdot\frac{L}{2} + \frac{L^2}{4}\right) = -\frac{17qL^4}{384EI} = -3.78 \text{ mm}$$

$$V(C) = \frac{-qL^2}{24EI}\left(6L^2 - 4L^2 + L^2\right) = -\frac{qL^4}{8EI} = -10.7 \text{ mm}$$

(b)(i)
$$\frac{d^2v}{dx^2} = -\frac{q}{2EI}\left(L^2 - 2Lx + x^2\right)$$
$$= -\frac{q}{2EI}(L-x)^2$$
$$= -\frac{1}{6000}(4-x)^2$$

$$V_{i+1} = V_i + \left(\frac{dv}{dx}\right)_i h \qquad \left(\frac{dv}{dx}\right)_{i+1} = \left(\frac{dv}{dx}\right)_i + \left(\frac{d^2v}{dx^2}\right)_i h$$

X	0	2	4
$\frac{dv}{dx}$	0	$\left(\frac{dv}{dx}\right)_0 + \left(\frac{d^2v}{dx^2}\right)_0 h = 0 - \frac{1}{375}\times 2 = -\frac{2}{375}$	$\left(\frac{dv}{dx}\right)_2 + \left(\frac{d^2v}{dx^2}\right)_2 h = -\frac{2}{375} + \left(-\frac{1}{1500}\times 2\right) = -\frac{1}{150}$
$\frac{d^2v}{dx^2}$	$-\frac{1}{6000}(4-0)^2 = -\frac{1}{375}$	$-\frac{1}{6000}(4-2)^2 = -\frac{1}{1500}$	$-\frac{1}{6000}(4-4)^2 = 0$
v(m)	0	$V_0 + \left(\frac{dv}{dx}\right)_0 h = 0 + 0 = 0$	$V_2 + \left(\frac{dv}{dx}\right)_2 h = 0 - \frac{2}{375}\times 2 = -\frac{4}{375}$

$V(C) = -\frac{4}{375}\times 1000 = -10.7$ mm	$V(B) = 0$ mm
$\varepsilon_t = \frac{-10.7 - (-10.7)}{-10.7}\times 100\% = 0$	$\varepsilon_t = \frac{-3.78 - 0}{-3.78}\times 100\% = 100\%$

(ii) $V_{i+\frac{1}{2}} = V_i + \left(\frac{dv}{dx}\right)_i \cdot \frac{h}{2}$ $\left(\frac{dv}{dx}\right)_{i+\frac{1}{2}} = \left(\frac{dv}{dx}\right)_i + \left(\frac{d^2v}{dx^2}\right)_i \frac{h}{2}$

$V_{i+1} = V_i + \left(\frac{dv}{dx}\right)_{i+\frac{1}{2}} h$ $\left(\frac{dv}{dx}\right)_{i+1} = \left(\frac{dv}{dx}\right)_i + \left(\frac{d^2v}{dx^2}\right)_{i+\frac{1}{2}} h$

x	0	1 mid-point	2	3 mid-point	4
$\frac{dv}{dx}$	0	$\left(\frac{dv}{dx}\right)_0 + \left(\frac{d^2v}{dx^2}\right)_0 \frac{h}{2} = -\frac{1}{375}$	$\left(\frac{dv}{dx}\right)_0 + \left(\frac{d^2v}{dx^2}\right)_1 h = -\frac{3}{1000}$	$\left(\frac{dv}{dx}\right)_2 + \left(\frac{d^2v}{dx^2}\right)_2 \cdot \frac{h}{2} = -\frac{11}{3000}$	$\left(\frac{dv}{dx}\right)_2 + \left(\frac{d^2v}{dx^2}\right)_3 h = -\frac{1}{300}$
$\frac{d^2v}{dx^2}$	$-\frac{1}{375}$	$-\frac{1}{6000}(4-1)^2 = -\frac{3}{2000}$	$-\frac{1}{6000}(4-2)^2 = -\frac{1}{1500}$	$-\frac{1}{6000}(4-3)^2 = -\frac{1}{6000}$	0
$v(m)$	0	$V_0 + \left(\frac{dv}{dx}\right)_0 \frac{h}{2} = 0$	$V_0 + \left(\frac{dv}{dx}\right)_1 h = -\frac{2}{375}$	$V_2 + \left(\frac{dv}{dx}\right)_2 \frac{h}{2} = -\frac{1}{120}$	$V_2 + \left(\frac{dv}{dx}\right)_3 h = -\frac{19}{1500}$

$v(C) = \frac{-19}{1500} \times 1000 = -12.7\ mm$

$\varepsilon_t = \frac{-10.7 - (-12.7)}{-10.7} \times 100\% = -18.7\%$

$v(B) = \frac{-2}{375} \times 1000 = -5.33\ mm$

$\varepsilon_t = \frac{-3.78 - (-5.33)}{-3.78} \times 100\% = -41\%$