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MT2004 - PYP 2021-2022 S2

Date: No:

1. a) 2030 → \$16M/greater

initial 2022 → \$5M, projected 2025 → \$8M

Consultant A: $\frac{dy}{dt} = ky$

$\int \frac{1}{y}\,dy = \int k\,dt$

$\ln|y| = kt + C$, C is a constant

$t = 0,\ y = 5$

$\ln|5| = k(0) + C$

$C = \ln 5$

$\ln|y| = kt + \ln 5$

$t = 3,\ y = 8$

$\ln|8| = 3k + \ln 5$

$3k = \ln|8| - \ln|5|$

$3k = \ln\left|\frac{8}{5}\right|$

$k = \frac{1}{3}\ln\left|\frac{8}{5}\right|$

$\ln|y| = \frac{1}{3}\ln\left|\frac{8}{5}\right|t + \ln 5$

$t = 8,\ y = ?$

$\ln|y| = \frac{1}{3}\ln\left|\frac{8}{5}\right|(8) + \ln 5$

$\ln|y| = 2.8627$

$y = e^{2.8627}$

$= 17.510$

≈ 17.5 (3.s.f) $>$ \$16M

Consultant B: $\frac{dy}{dt} = ky(M-y)$

$M = 25$

$\frac{dy}{dt} = ky(25-y)$

$\frac{1}{y(25-y)} = \frac{A}{y} + \frac{B}{25-y}$

$1 = A(25-y) + By$

$y = 0$ | $y = 25$

$1 = 25A + 0$ | $1 = 25B$

$A = \frac{1}{25}$ | $B = \frac{1}{25}$

$\int \frac{1}{y(25-y)}\,dy = \int k\,dt$

$\cancel{\int \frac{1}{25y} + \frac{1}{25(25-y)}\,dy = \int k\,dt}$

$\frac{1}{25}\int\left(\frac{1}{y} + \frac{1}{25-y}\right)dy = \int k\,dt$

$\frac{1}{25}\left(\ln y + \ln|25-y|(-1)\right) = kt + C$

$\frac{1}{25}\left(\ln y - \ln|25-y|\right) = kt + C$

$t = 0,\ y = 5$

$\frac{1}{25}(\ln 5 - \ln 20) = k(0) + C$

$C = \frac{1}{25}\ln\frac{1}{4}$

$\frac{1}{25}\left(\ln\frac{y}{25-y}\right) = kt + \frac{1}{25}\ln\frac{1}{4}$

$t = 3,\ y = 8$

$\frac{1}{25}\left(\ln\frac{8}{25-8}\right) = k(3) + \frac{1}{25}\ln\frac{1}{4}$

$\ln\frac{8}{17} - \ln\frac{1}{4} = 75k$

$\ln\frac{32}{17} = 75k$

$k = \frac{1}{75}\ln\frac{32}{17}$

$\frac{1}{25}\ln\left(\frac{y}{25-y}\right) = \frac{1}{75}\ln\left(\frac{32}{17}\right)t + \frac{1}{25}\ln\frac{1}{4}$

$t = 8,\ y = ?$

$\ln\frac{y}{25-y} = 0.3004$

$\frac{y}{25-y} = e^{0.3004}$

$= 1.3504$

$y = 1.3504(25-y)$

$2.350442y = 33.7610$

$y = 14.363$

$\approx 14.4 \quad (< \$16M)$

$\therefore$ For consultant A, I will buy as the 2030 projection is \$17.5M > \$16.5M

For consultant B, I will not buy as the 2030 projection is \$14.4M < \$16.5M

1.b) Consultant B gives a more realistic revenue projection as a logistic growth model. is used compared to consultant A who uses unlimited growth model. Consultant B use of logistic growth model takes into account the limited amount of resources in the real world. There would be a limit to how much the revenue can increase before remaining at a constant.

2. a) $Ax = b$

$$\begin{pmatrix} 3 & 6 & 5 \\ 2 & 1 & -2 \\ 5 & 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 3 & 6 & 5 & 8 \\ 2 & 1 & -2 & 6 \\ 5 & 4 & -3 & 2 \end{array}\right)$$

$R_1 \div 3 \to R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 5/3 & 8/3 \\ 2 & 1 & -2 & 6 \\ 5 & 4 & -3 & 2 \end{array}\right)$$

$R_2 - 2R_1 \to R_2 \quad R_3 - 5R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 5/3 & 8/3 \\ 0 & -3 & -16/9 & 2/9 \\ 0 & -6 & -34/3 & -34/3 \end{array}\right)$$

$R_2 \div -3 \to R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 5/3 & 8/3 \\ 0 & 1 & 16/9 & -2/9 \\ 0 & -6 & -34/3 & -34/3 \end{array}\right)$$

$R_1 - 2R_2 \to R_1 \quad R_3 + 6R_2 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -17/9 & 28/9 \\ 0 & 1 & 16/9 & -2/9 \\ 0 & 0 & -2/3 & -38/3 \end{array}\right)$$

$R_3 \div -\frac{2}{3} \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -17/9 & 28/9 \\ 0 & 1 & 16/9 & -2/9 \\ 0 & 0 & 1 & 19 \end{array}\right)$$

$R_1 + \frac{17}{9}R_3 \to R_1 \quad R_2 - \frac{16}{9}R_3 \to R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & -34 \\ 0 & 0 & 1 & 19 \end{array}\right)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 39 \\ -34 \\ 19 \end{pmatrix}$$

2. b)

$$24\frac{dy}{dy} + 12\frac{d^2y}{dx^2} - 36y = 0$$

$$12\frac{d^2y}{dx^2} + 24\frac{dy}{dx} - 36y = 0$$

characteristic eqn (CE) : $12r^2 + 24r - 36 = 0$

$12(r-1)(r+3) = 0$

$r_1 = 1$ or $r_2 = -3$

$r_1 \neq r_2 \rightarrow$ distinct

$$y = C_1 e^{x} + C_2 e^{-3x}$$

$x = 0,\ y = 4$

$4 = C_1(1) + C_2(1)$

$C_1 + C_2 = 4$

$C_1 = 4 - C_2$

$x = \ln 2,\ y = 8$

$8 = C_1 e^{\ln 2} + C_2 e^{-3\ln 2}$

$8 = 2C_1 + \frac{1}{8}C_2$

$8 = 2(4 - C_2) + \frac{1}{8}C_2$

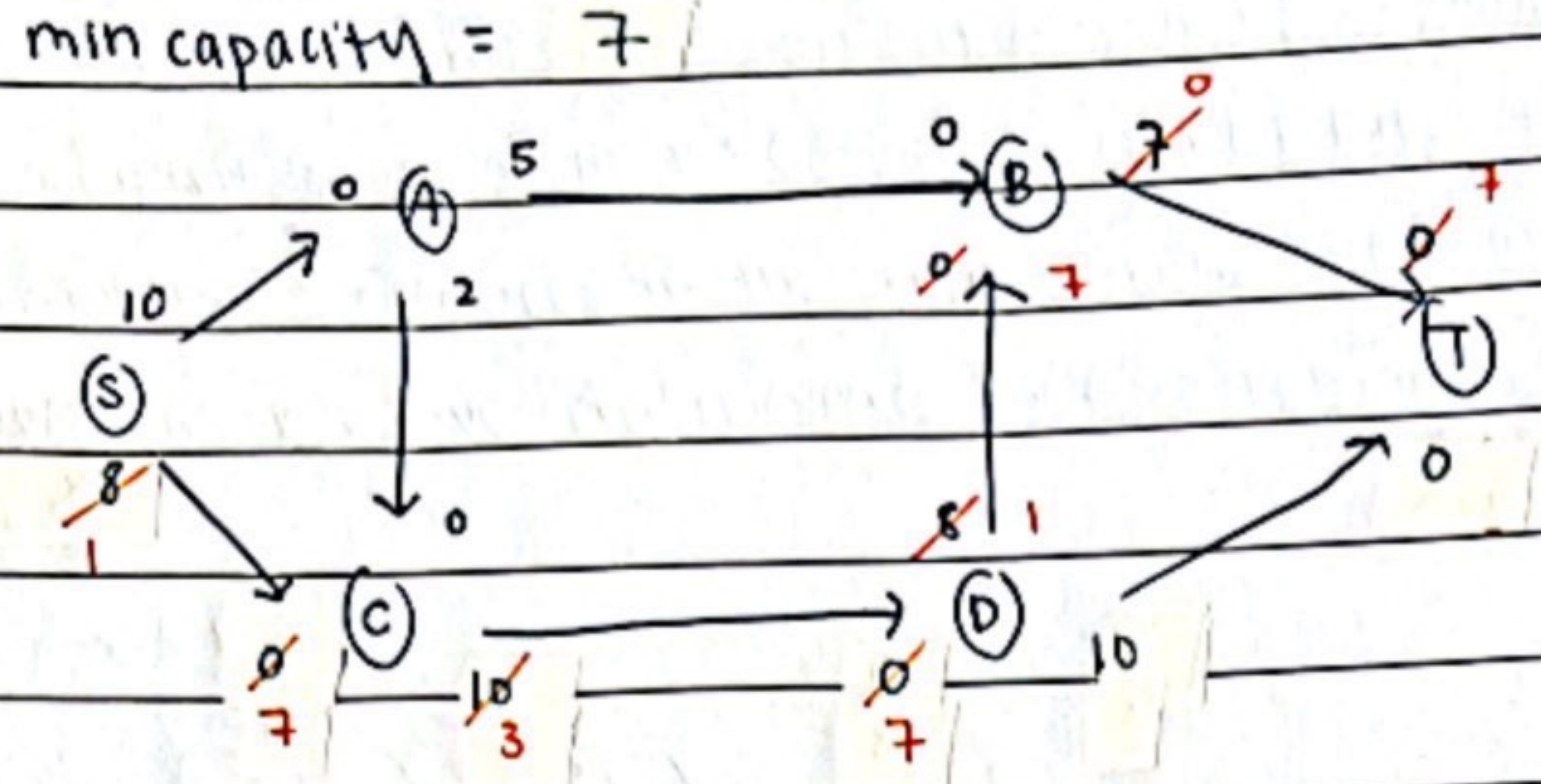
$0 = -\frac{15}{8}C_2$

$C_2 = 0$

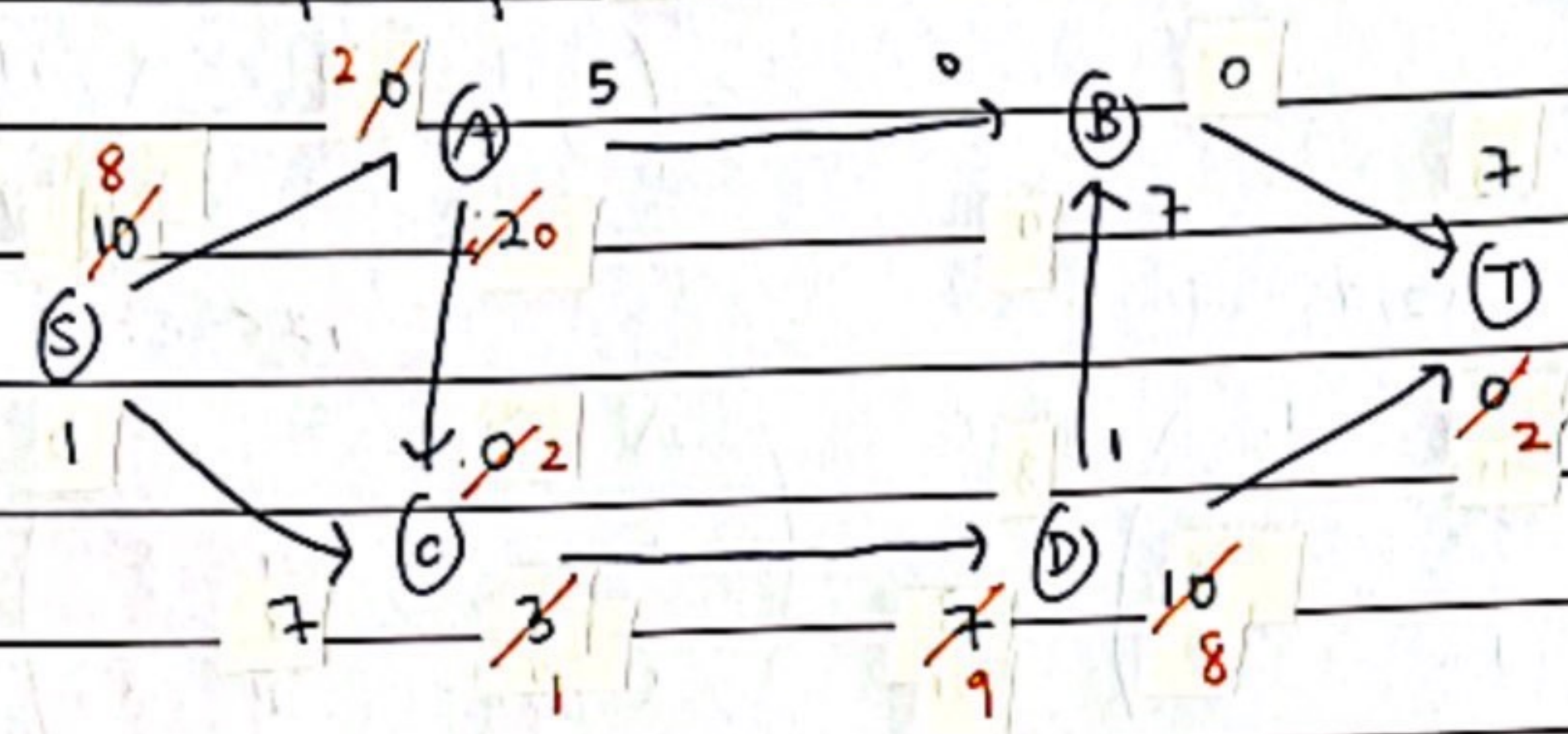
$C_1 = 4$

$\therefore\ y = 4e^{x}$

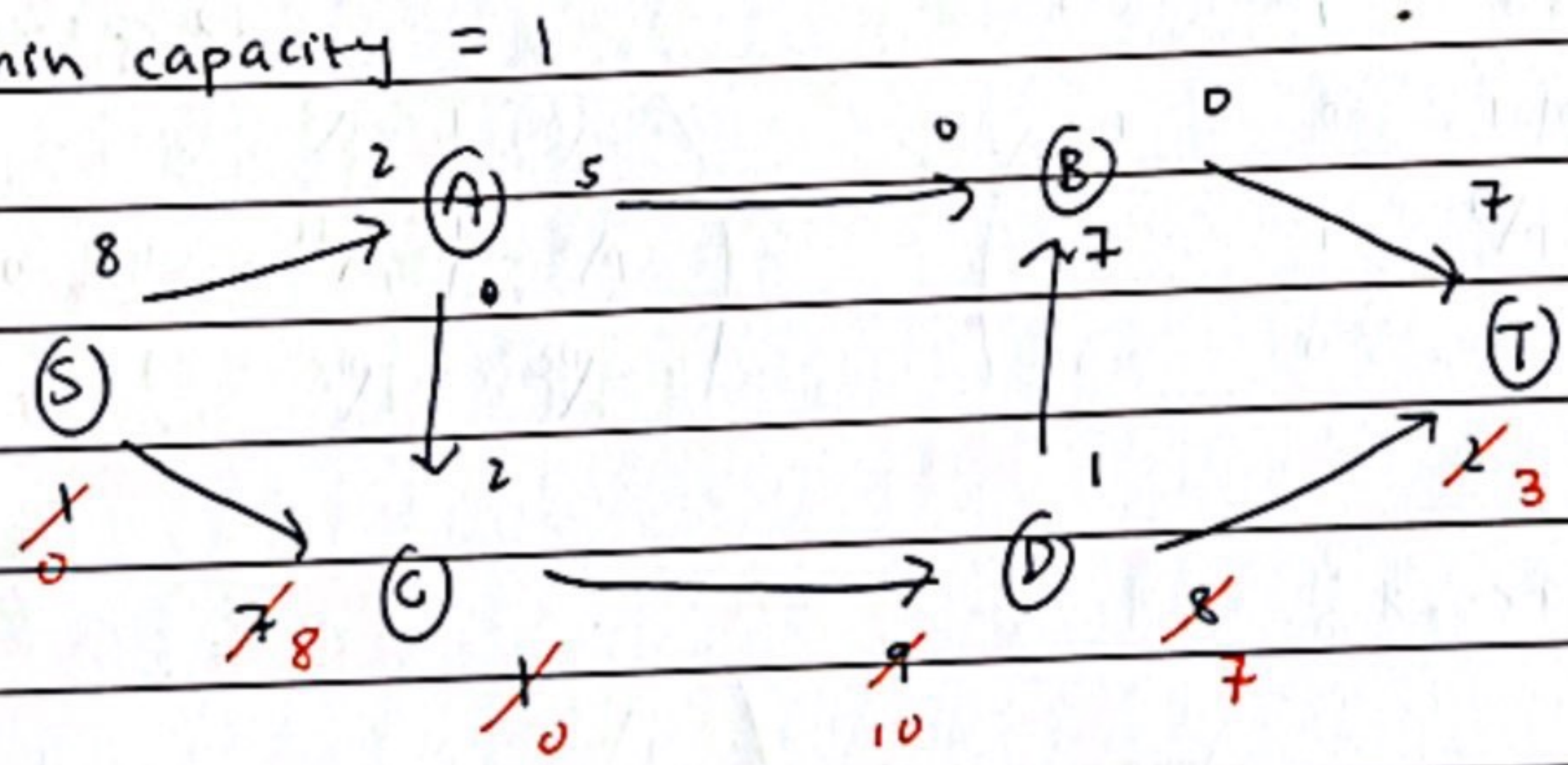
3. • augmenting path: S → C – D → B → T
min capacity = 7



• augmenting path: S → A → C → D – T
min capacity = 2



• augmenting path: S → C → D → T
min capacity = 1



no more augmenting path
∴ max flow = 8+2 = 10 //

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4. a) max total profit, $P = 18(310)(x_{11} + x_{12} + x_{13}) + 15(380)(x_{21} + x_{22} + x_{23})$
$+ 23(350)(x_{31} + x_{32} + x_{33}) + 12(285)(x_{41} + x_{42} + x_{43})$ //

$x_{ij} > 0$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$

s.t. $18x_{11} + 15x_{21} + 23x_{31} + 12x_{41} \leq 10$

$18x_{12} + 15x_{22} + 23x_{32} + 12x_{42} \leq 16$

$18x_{13} + 15x_{23} + 23x_{33} + 12x_{43} \leq 8$

$480(18x_{11}) + 650(15x_{21}) + 580(23x_{31}) + 390(12x_{41}) \leq 6800$

$480(18x_{12}) + 650(15x_{22}) + 580(23x_{32}) + 390(12x_{42}) \leq 8700$

fc $480(18x_{13}) + 650(15x_{23}) + 580(23x_{33}) + 390(12x_{43}) \leq 5300$

add surplus , y_3 and x_4

add artifical variable , x_5 and x_6

4. b) min $z = x_1 + x_2$

s.t $2x_1 + 4x_2 \geq 4$

$x_1 + 6x_2 \geq 5$

$x_1 , x_2 \geq 0$

max $f(-z) = x_1 + x_2 - Mx_5 - Mx_6$

$2x_1 + 4x_2 - x_3 + x_5 = 4$

$x_1 + 6x_2 - x_4 + x_6 = 5$

basic variables	eqn	f(-z)	x_1	x_2	x_3	x_4	x_5	x_6	RHS	ratio
-z	(0)	1	-1	-1	0	0	M	M	0	(0) - (1)M → (0)
x_3	(1)	0	1	4	-1	0	1	0	4	
x_4	(2)	0	1	6	0	-1	0	1	5	

-z	(0)	1	M-1	4M-1	M	0	0	M	-4M	(0) - (2)M → (0)
x_3	(1)	0	1	4	-1	0	1	0	4	
x_4	(2)	0	1	6	0	-1	0	1	5	

				↓							
-z	(0)	1	-1	-2M-1	M	M	0	0	-9M		(0) + (2M-1)(2)N → (0)
x_3	(1)	0	1	4	-1	0	1	0	4	1	(1) - 4(2)N → (1)
x_4	(2)	0	1	6	0	-1	0	1	5	5/6	(2) ÷ 6 → (2)N

inital BFS = { 0, 0, 4, 5, 0, 0 }

-z	(0)	1	$\frac{1}{3}M-\frac{7}{6}$	0	M	$\frac{2}{3}M+\frac{1}{6}$	0	$\frac{1}{3}M-\frac{1}{6}$	$-\frac{22}{6}M-\frac{5}{6}$
x_3	(1)	0	$\frac{1}{3}$	0	-1	$\frac{2}{3}$	1	$-2/3$	$2/3$
x_2	(2)	0	$\frac{1}{6}$	1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	$5/6$

next BFS = { 0, $\frac{5}{6}$, $\frac{2}{3}$, 0 , 0 , 0 }

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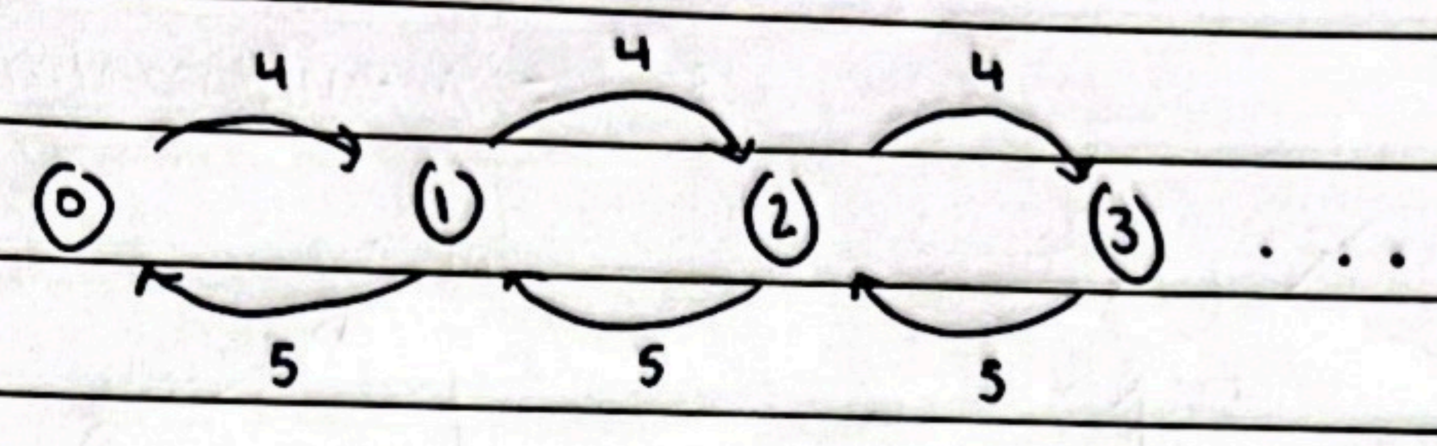
5. a) M/M/1

$\rho = \frac{\lambda}{s\mu} = \frac{4}{5}$



$P_1 = \frac{4}{5} P_0$

$P_2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25} P_0$

$P_3 = \frac{64}{125} P_0$

$P_0 \left(1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \dots\right) = 1$

$P_0 = 1 - \frac{4}{5} = \frac{1}{5}$

P(more than 2 custumers queving) $= 1 - P_0 - P_1$

$= 1 - \frac{1}{5} - \frac{4}{5}\left(\frac{1}{5}\right)$

$= 0.64$

5. b) $L = \sum_{n=0}^{k} nP_n = \frac{1-p}{1-p^{k+1}} \sum_{n=0}^{k} np^n$

$$= \frac{p(1-p)}{1-p^{k+1}} \sum np^{n-1}$$

$$= \frac{p(1-p)}{1-p^{k+1}} \sum \frac{d}{dp} p^n$$

$$= \frac{p(1-p)}{1-p^{k+1}} \frac{d}{dp} \sum p^n$$

$$= \frac{p(1-p)}{1-p^{k+1}} \frac{d}{dp} \left(\frac{1-p^{k+1}}{1-p} \right)$$

$$= \frac{p(1-p)}{1-p^{k+1}} \left[\frac{(1-p)(1-p^{k+1})(-k-1) - (1-p^{k+1})(-1)}{(1-p)^2} \right]$$

$$= \frac{p(1-p^{k+1}) - (k+1)(p^{k+1})(1-p)}{(1-p)(1-p^{k+1})}$$

$$= \frac{p(1-p^{k+1})}{(1-p)(1-p^{k+1})} - \frac{(k+1)p^{k+1}(1-p)}{(1-p^{k+1})(1-p)}$$

$$= \frac{p}{1-p} - \frac{(k+1)p^{k+1}}{1-p^{k+1}} \quad //$$

$L_q = \sum (n-1)P_n = \sum nP_n - P_n$

$$= \sum nP_n - \sum P_n$$

$$= L - (1-P_0) \quad //$$