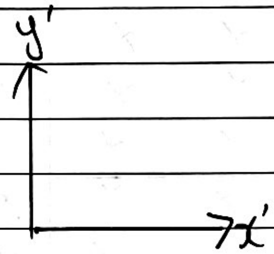
CV4101 AY 18/19 Date: No:

1a) 2DOF: u_4', v_4'



b) Unit: [kN ; m]

$E = 200 \times 10^6$ kN/m²
$L_1 = L_2 = L_3 = 3$ m
$A_1 = A_2 = A_3 = 3 \times 10^{-3}$ m²

member 1 (1-4) ($C_1 = 1$, $S_1 = 0$)

$$\frac{EA_1}{L_1} = 200000 = \frac{EA_2}{L_2} = \frac{EA_3}{L_3}$$

$$K_1' = \begin{bmatrix} 200000 & 0 & -200000 & 0 \\ 0 & 0 & 0 & 0 \\ -200000 & 0 & 200000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

member 2 (2-4) ($C_2 = 0$, $S_2 = 1$)

$$K_2' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 200000 & 0 & -200000 \\ 0 & 0 & 0 & 0 \\ 0 & -200000 & 0 & 200000 \end{bmatrix}$$

member 3 (3-4) ($C_3 = 0.5$, $S_3 = \frac{1}{2}\sqrt{3}$)

$$K_2' = \begin{bmatrix} 50000 & 50000\sqrt{3} & -50000 & -50000\sqrt{3} \\ 50000\sqrt{3} & 150000 & -50000\sqrt{3} & -150000 \\ -50000 & -50000\sqrt{3} & 50000 & 50000\sqrt{3} \\ -50000\sqrt{3} & -150000 & 50000\sqrt{3} & 150000 \end{bmatrix}$$

Assemble:

$$\begin{bmatrix} R_{1x}' \\ R_{1y}' \\ R_{2x}' \\ R_{2y}' \\ R_{3x}' \\ R_{3y}' \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & 250000 & 50000\sqrt{3} \\ \times & \times & \times & \times & \times & \times & 50000\sqrt{3} & 350000 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4' \\ v_4' \end{bmatrix}$$

$$\begin{bmatrix} 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 250000 & 50000\sqrt{3} \\ 50000\sqrt{3} & 350000 \end{bmatrix} \begin{bmatrix} u_4' \\ v_4' \end{bmatrix}$$

$u_4' = 6.585 \times 10^{-4}\,m = 0.6585\,mm$

$v_4' = 4.085 \times 10^{-4}\,m = 0.4085\,mm$

(i) $L_3 \to \infty$; $\frac{AE_3}{L_3} = 0$

$$\begin{bmatrix} 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 200000 + \frac{1}{4}\left(\frac{AE_3}{L_3}\right) & \frac{\sqrt{3}}{4}\frac{AE_3}{L_3} \\ \frac{\sqrt{3}}{4}\frac{AE_3}{L_3} & 200000 + \frac{3}{4}\frac{AE_3}{L_3} \end{bmatrix} \begin{bmatrix} u_4' \\ v_4' \end{bmatrix}$$

$$\begin{bmatrix} 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 200000 & 0 \\ 0 & 200000 \end{bmatrix} \begin{bmatrix} u_4' \\ v_4' \end{bmatrix}$$

$u_4' = v_4' = 10^{-3}\,m = 1\,mm$

ii) $L_3 \to 0$; $\frac{AE_3}{L_3} \to \infty$

$u_4' = v_4' = 0$ (member 3-4 does not have any axial deformation)

2a) 2DOF: v_2' ; θ_2'

b) Unit: [kN ; m]

$E = 200\ kN/mm^2 = 200 \times 10^6\ kN/m^2$ $w = 16$

$L_1 = L_2 = 2m = L$

$I_1 = I_2 = 4 \times 10^{-6}\ m^4 = I$

member 1

$\frac{EI}{L^3} = 100$ $\frac{EI}{L^2} = 200$ $\frac{EI}{L} = 400$

$$K_1' = \left[\begin{array}{cc|cc} 1200 & 1200 & -1200 & 1200 \\ 1200 & 1200 & -1200 & 800 \\ \hline -1200 & -1200 & 1200 & -1200 \\ 1200 & 800 & -1200 & 1600 \end{array}\right]$$

member 2

$$K_{m2}' = \left[\begin{array}{cc|cc} 300 & 600 & -300 & 0 \\ 600 & 1200 & -600 & 0 \\ \hline -300 & -600 & 300 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Force vector

$$F_{02}' = \begin{bmatrix} 5wL/8 \\ wL^2/8 \\ 3wL/8 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ 12 \\ 0 \end{bmatrix}$$

Assemble

$$\begin{bmatrix} V_1' \\ M_1' \\ 10 \\ 0 \\ V_3' \\ 0 \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & 1500 & -600 & \times & \times \\ \times & \times & -600 & 2800 & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_2' \\ \theta_2' \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0+20 \\ 0+8 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1500 & -600 \\ -600 & 2800 \end{bmatrix} \begin{bmatrix} V_2' \\ \theta_2' \end{bmatrix} + \begin{bmatrix} 20 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} -10 \\ -8 \end{bmatrix} = \begin{bmatrix} 1500 & -600 \\ -600 & 2800 \end{bmatrix} \begin{bmatrix} V_2' \\ \theta_2' \end{bmatrix}$$

$V_2' = -8.542 \text{ mm}$
$\theta_2' = -4.6875 \text{ mm}$

c) $V_2' = -8.542 \times 10^{-3} \text{ m}$

$$F_{02}' = \begin{bmatrix} 11F/16 \\ 3FL/16 \\ 5F/16 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{16}P \\ \frac{3}{8}P \\ \frac{5}{16}P \\ 0 \end{bmatrix}$$

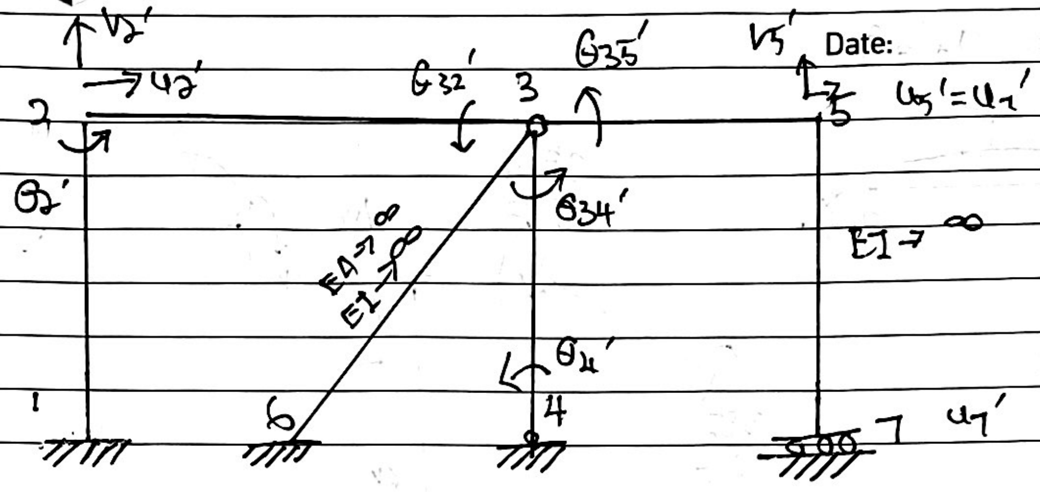
$$\begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1500 & -600 \\ -600 & 2800 \end{bmatrix} \begin{bmatrix} -8.542 \times 10^{-3} \\ \theta_2' \end{bmatrix} + \begin{bmatrix} 11/16\,P \\ 3/8\,P \end{bmatrix}$$

$10 = -12.813 - 600\,\theta_2' + \frac{11}{16}P$
$22.813 = -600\,\theta_2' + \frac{11}{16}P \quad -(1)$

$0 = 5.1252 + 2800\,\theta_2' + \frac{3}{8}P$
$-5.1252 = 2800\,\theta_2' + \frac{3}{8}P \quad -(2)$

$\theta_2' = -5.613 \times 10^{-3}$
$P = 28.28 \text{ kN}$

3a)



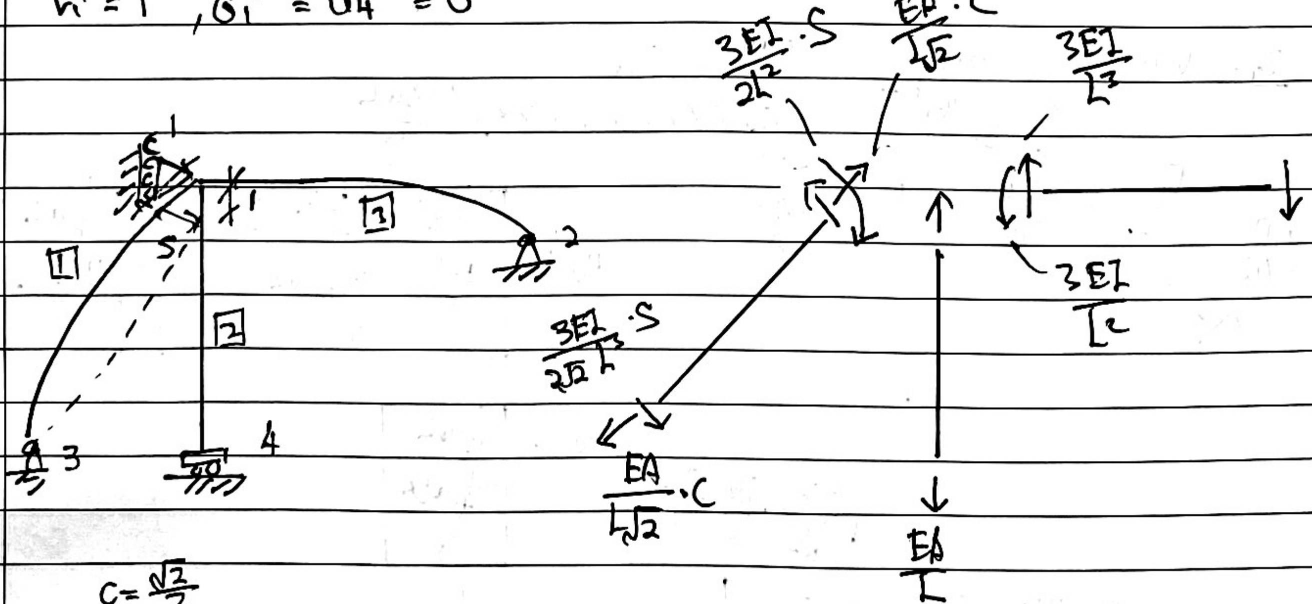
Joint 3: DOF − θ_{32}'
− θ_{34}'
− θ_{35}'

u_3' and v_3' are restricted by member 3-6 due to having $EA \to \infty$. θ_{36}' is restricted by member 3-6 due to having $EI \to \infty$. As joint 3 is an internal pin, member 2-3, 3-4 & 3-5 are unaffected by member 3-6 properties. This means that there will be rotation θ_{32}', θ_{34}' and θ_{35}'.

b) $P = \sqrt{2}\,wL$

DOF: v_1', θ_1', u_4'

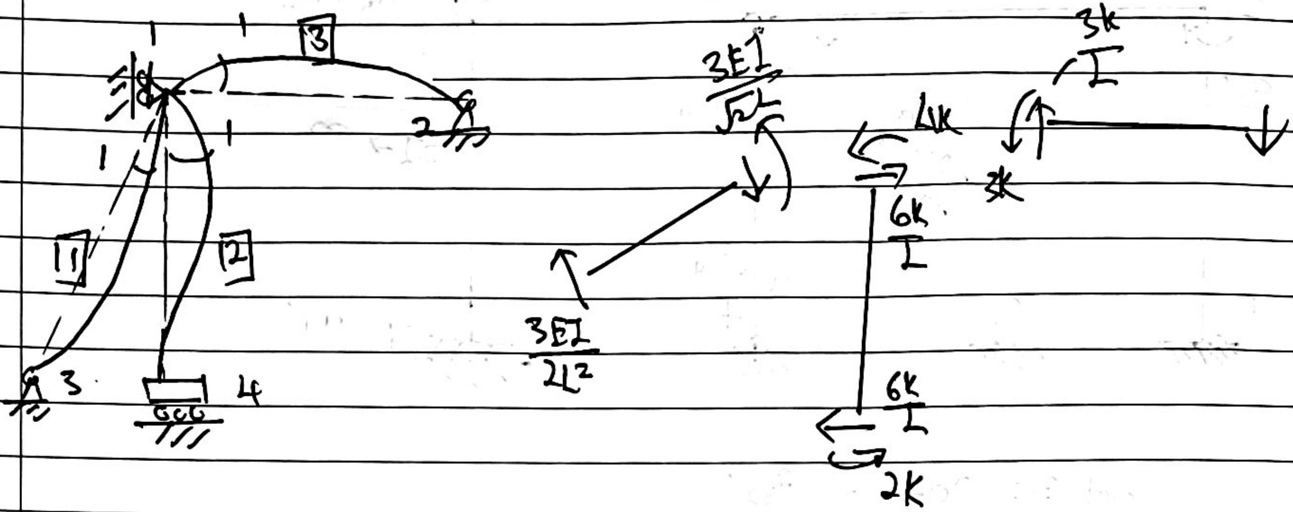
$v_1' = 1$, $\theta_1' = u_4' = 0$



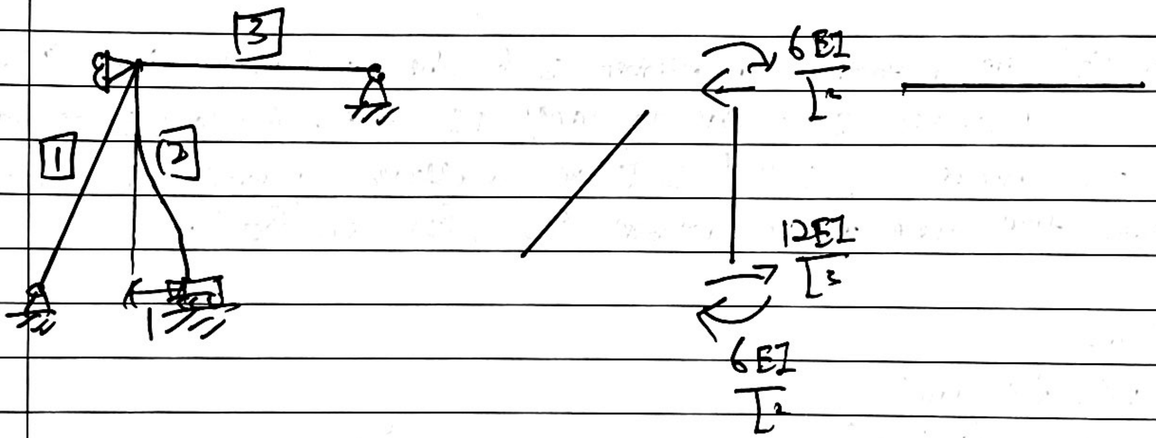
$C = \frac{\sqrt{2}}{2}$

$S = \sqrt{2}/2$

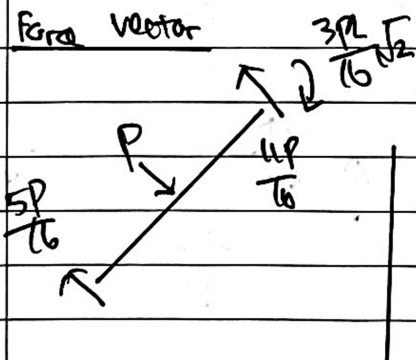
$\theta_1 = 1$, $v_1' = u_4' = 0$

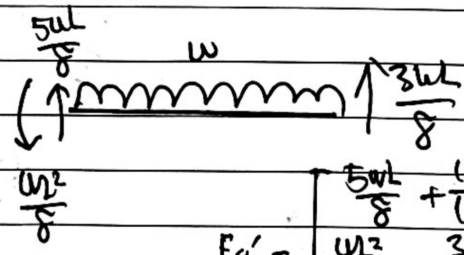


$u_4' = 1$, $v_1' = \theta_1' = 0$



Force vector





$$F_0' = \begin{bmatrix} \frac{5wL}{8} + \frac{11}{16}P\cdot\frac{\sqrt{2}}{2} \\ \frac{wL^2}{8} - \frac{3PL}{16}\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{8}wL + \frac{11}{16}wL \\ \frac{wL^2}{8} - \frac{3}{8}wL^2 \\ 0 \end{bmatrix}$$

$$F_0' = \begin{bmatrix} \frac{21}{16}wL \\ -\frac{1}{4}wL^2 \\ 0 \end{bmatrix}$$

$$+\frac{EA}{L\sqrt{2}}\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} + \frac{3EI}{L^3} + \frac{3EI}{2\sqrt{2}L^3}\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} & \frac{3EI}{L^2} - \frac{3EI}{2L^2}\cdot\frac{\sqrt{2}}{2} & 0 \\ \frac{3EI}{L^2} - \frac{3EI}{2L^2}\cdot\frac{\sqrt{2}}{2} & \frac{7EI}{L} + \frac{3}{\sqrt{2}}\frac{EI}{L} & -6EI/L^2 \\ 0 & -\frac{6EI}{L^2} & 12EI/L^3 \end{bmatrix} + \begin{bmatrix} \frac{21}{16}wL \\ -\frac{1}{4}wL^2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{21}{16}wL \\ \frac{1}{4}wL^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(1+\frac{\sqrt{2}}{4}\right)\frac{EA}{L} + \frac{EI}{L^3}\left(3+\frac{3}{8}\sqrt{2}\right) & \frac{EI}{L^2}\left(3-\frac{3}{4}\sqrt{2}\right) & 0 \\ \frac{EI}{L^2}\left(3-\frac{3}{4}\sqrt{2}\right) & \frac{EI}{L}\left(7+\frac{3}{2}\sqrt{2}\right) & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} \end{bmatrix} \begin{bmatrix} v_1' \\ \theta_1' \\ u_4' \end{bmatrix}$$

$A = \frac{I}{L^2}$

$$-\frac{21}{16}wL = \left(1+\frac{\sqrt{2}}{4}\right)\frac{EI}{L^3} + \frac{EI}{L^3}\left(3+\frac{3}{8}\sqrt{2}\right)v' + \frac{EI}{L}\left(3-\frac{3}{4}\sqrt{2}\right)\theta_1'$$

$$-\frac{21}{16}wL^2 = \frac{EI}{L^2}\left(4+\frac{5}{8}\sqrt{2}\right)v_1' + \frac{EI}{L}\left(3-\frac{3}{4}\sqrt{2}\right)\theta_1'$$

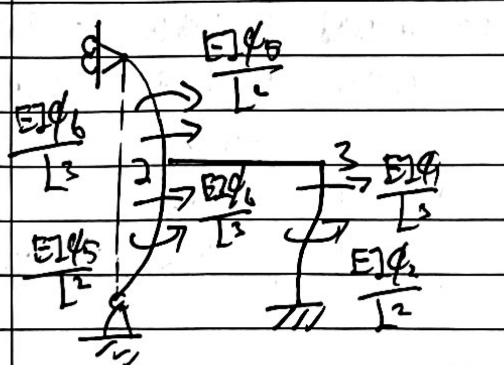
$v' = -0.3239\,\frac{wL^4}{EI}$

$\theta_1' = 0.1441\,\frac{wL^3}{EI}$

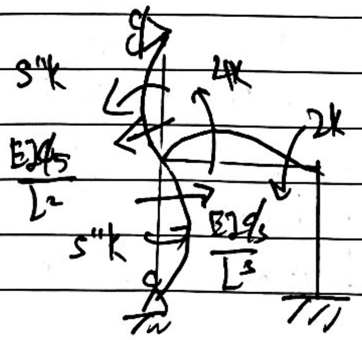
$u_4' = 0.07206\,\frac{wL^4}{EI}$

4a) DOF = u_2', θ_2', θ_3'

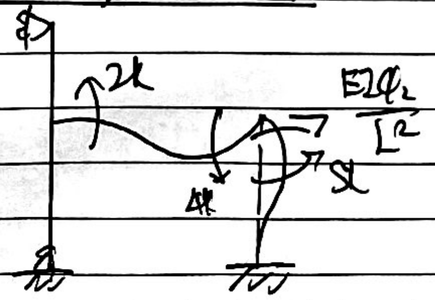
$u_2' = 1$, $\theta_2' = \theta_3' = 0$



$\theta_2' = 1$, $u_2' = \theta_3' = 0$



$\theta_3' = 1$ $u_2' = \theta_2' = 0$



$$K_{ff}' = \begin{bmatrix} \frac{2EI\phi_1}{L^3} + \frac{EI\phi_1}{L^3} & 0 & \frac{EI\phi_2}{L^2} \\ 0 & (2s''+4)k & 2k \\ \frac{EI\phi_2}{L^2} & 2k & (s+4)k \end{bmatrix}$$

$$|K_{ff}'| = 0$$

$$\frac{EI}{L^3}(2\phi_6+\phi_1) - (2s''+4)k \cdot (s+4)k - \left(\frac{EI\phi_2}{L^2}\right)^2 \cdot (2s''+4)k - 4k^2 \cdot \frac{EI}{L^3}(2\phi_6+\phi_1)$$

$$(2\phi_6+\phi_1)(2s''+4)(s+4) - \phi_2^2(2s''+4) - 4(2\phi_6+\phi_1) = 0$$

$$(2\phi_6+\phi_1)\left[(2s''+4)(s+4) - 4\right] - \phi_2^2(2s''+4) = 0$$

$$\phi_1 = 2s(1+c) - \pi^2\rho$$

$$\phi_2 = s(1+c)$$

$$\phi_6 = s'' - \pi^2\rho$$

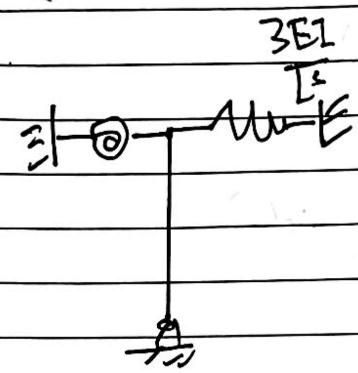
$0 < \rho < 2.05$

ρ	s	c	s''	ϕ_1	ϕ_2	ϕ_6	$f(x)$
1	2.4674	1	0	-4.4×10^{-6}	4.9348	-9.8696	-529.097
0.2	3.7297	0.555	2.5808	9.625	5.7997	0.6069	416.05
0.6	3.1402	0.7136	1.5413	4.84	5.381	-4.38	-387.64
0.4	3.4439	0.6262	2.1021	7.239	5.594	-1.846	-543
0.36	3.5024	0.6090	2.2035	7.824	5.6866	-1.3491	30.684

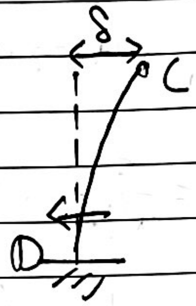
$$\rho = 0.36 + \frac{0 - 30.684}{-543 - 30.684} \times 0.04$$

$$\rho = 0.3744$$

b)

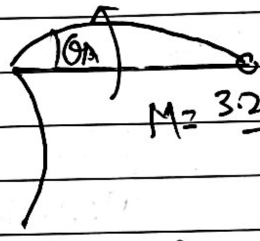


Translational spring

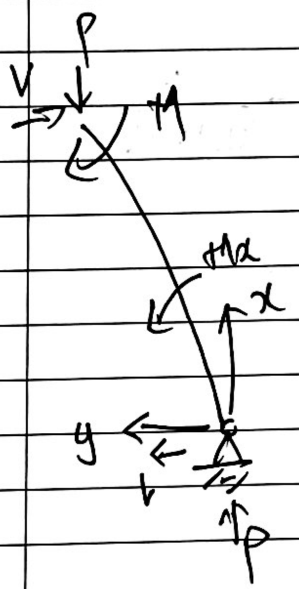


$F = \frac{3EI}{L^3} \cdot \delta$ $k_T = \frac{3EI}{L^3}$

Rotational spring



$M = \frac{3 \cdot 2EI}{L} \theta_A$ $k_R = \frac{6EI}{L}$



$V = \frac{3EI}{L^3} \cdot v(L)$

$M = \frac{6EI}{L} \theta_A$

$M_x + Pv = Vx$

$EIv'' + Pv = \frac{3EI}{L^3} \cdot v(L) x$

$v'' + \omega^2 v = \frac{3}{L^3} v(L) x$

2nd ODE

$\omega^2 \cdot c_1 x = \frac{3}{L^3} v(L) \cdot x$

$c_1 = \frac{3}{\omega^2 L^3} \cdot v(L)$

$v = A \sin \omega x + B \cos \omega x + \frac{3}{\omega^2 L^3} \cdot v(L) x$

$v' = A \cos \omega x - B\omega \sin \omega x + \frac{3}{\omega^2 L^3} v(L)$

$v'' = -A\omega^2 \sin \omega x - B\omega^2 \cos \omega x$

Boundary condition

$v(0) = 0 \rightarrow B = 0$

$v''(L) = -\frac{M}{EI} \rightarrow -\frac{M}{EI} = -A\omega^2 \sin \omega L$

$\frac{6\theta_A}{L} = A\omega^2 \sin \omega L$

$A = \frac{6\theta_A}{\omega^2 L \sin \omega L}$

$$v'(L) = \theta_A \Rightarrow \theta_A = A\omega \cos\omega x + \frac{3}{\omega^2 L^3} v(L)$$

$$\theta_A = \frac{6\theta_A}{\omega^2 L \sin\omega L} \cdot \omega \cos\omega L + \frac{3}{\omega^2 L^3} \cdot v(L)$$

$$\theta_A - \frac{6\theta_A}{\omega L \cdot \tan\omega L} = \frac{3}{\omega^2 L^3} \cdot v(L)$$

$$v(L) = \frac{\omega^2 L^3}{3} \theta_A \left(1 - \frac{6}{\omega L \tan\omega L} \right)$$

$$v(L) = A \sin\omega L + \frac{3}{\omega^2 L^3} v(L) \cdot L$$

$$\frac{\omega L^3}{3} \theta_A \cdot \left(1 - \frac{6}{\omega L \tan\omega L} \right) = \frac{6\theta_A}{\omega^2 L \sin\omega L} \cdot \sin\omega L + \left(\theta_A - \frac{6\theta_A}{\omega L \tan\omega L} \right) \cdot L$$

$$\omega \times \left[\frac{\omega^2 L^3}{3} \left(1 - \frac{6}{\omega L \tan\omega L} \right) \right] = \left[\frac{6}{\omega^2 L} + L - \frac{6}{\omega \tan\omega L} \right] \times \omega$$

$$\varphi = \omega L$$

$$\underbrace{\frac{\varphi^3}{3} \left(1 - \frac{6}{\varphi \tan\varphi} \right) = \frac{6}{\varphi} + \varphi - \frac{6}{\tan\varphi}}_{\times 3\varphi}$$

$$\varphi^4 \left(1 - \frac{6}{\varphi} \cot\varphi \right) = 18 + 3\varphi^2 - 18\varphi \cot\varphi$$

$$\varphi^4 - 6\varphi^3 \cdot \cot\varphi - 3\varphi^2 + 18\varphi \cot\varphi = 18$$

$$\varphi^2(\varphi^2 - 3) - 6\varphi \cot\varphi \, (\varphi^2 - 3) = 18$$

$$(\varphi^2 - 6\varphi \cot\varphi)(\varphi^2 - 3) - 18 = 0$$