Suggested Solutions by Min Myat Thant Oo   CV2019 AY22/23

1. (a) $x = [x_1, x_2, x_3]^T$

$$\underset{3\times3}{\begin{bmatrix} -3 & 4 & 1 \\ 3 & -4 & 2 \\ 3 & 2 & 5 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} -16 \\ 19 \\ 10 \end{bmatrix}}$$

$$\text{Augmented Matrix} = \underset{3\times4}{\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 3 & -4 & 2 & 19 \\ 3 & 2 & 5 & 10 \end{array}\right]}$$

$R1+R2 \to R2$
$R1+R3 \to R3$

$$\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 0 & 0 & 3 & 3 \\ 0 & 6 & 6 & -6 \end{array}\right]$$

Swap R2 & R3

$$\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 0 & 6 & 6 & -6 \\ 0 & 0 & 3 & 3 \end{array}\right] \text{ (row echelon form)}$$

$3x_3 = 3$
$x_3 = 1$

$6x_2 + 6x_3 = -6$
$6x_2 = -6 - 6(1)$
$= -12$
$\therefore x_2 = -2$

$-3x_1 + 4x_2 + x_3 = -16$
$-3x_1 + 4(-2) + 1 = -16$
$-3x_1 = -9$
$x_1 = 3$

$$x = [x_1, x_2, x_3]^T_{1\times3}$$
$$= [3 \;\; -2 \;\; 1]^T$$
$$= \underset{3\times1}{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}$$

Use GC to check

$$\text{rref} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_1 = 3$
$x_2 = -2$
$x_3 = 1$

$$\text{ref} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} & \frac{16}{3} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_3 = 1$
$x_2 + x_3 = -1$
$\therefore x_2 = -2$
$x_1 - \frac{4}{3}x_2 - \frac{1}{3}x_3 = \frac{16}{3}$
$x_1 - \frac{4}{3}(-2) - \frac{1}{3}(1) = \frac{16}{3}$
$\therefore x_1 = 3$

(b)

$$\text{Augmented Matrix} = \underset{5\times4}{\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 3 & -4 & 2 & 19 \\ 3 & 2 & 5 & 10 \\ -6 & 8 & 2 & a \\ 0 & 6 & 6 & b \end{array}\right]}$$

repeat same steps in (a)
just copy over

$$\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 0 & 6 & 6 & -6 \\ 0 & 0 & 3 & 3 \\ -6 & 8 & 2 & a \\ 0 & 6 & 6 & b \end{array}\right]$$

6 -8 -2 32

$R5 - R2 \to R5$
$R1 \times (-2) + R4 \to R4$

$$\left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 0 & 6 & 6 & -6 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & a+32 \\ 0 & 0 & \underbrace{0}_{P} & b+6 \end{array}\right] \xrightarrow{\text{Swap R4 \& R5}} \left[\begin{array}{ccc|c} -3 & 4 & 1 & -16 \\ 0 & 6 & 6 & -6 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & b+6 \\ 0 & 0 & \underbrace{0}_{P} & a+32 \end{array}\right]$$

Recall for solution to exist & be unique, $P \neq 0$

$$\begin{bmatrix} a & b & c & d & e \\ 0 & f & g & h & i \\ 0 & 0 & j & k & l \\ 0 & 0 & 0 & P & q \end{bmatrix}$$

a, f, j are non-zero

Since $P = 0$ for both cases above,
it is impossible for x to exist and be unique.

Check GC with any arbitrary a & b and you get error for rref and ref.

(c) $D = (A^{-1}+B)^4 C^{-7}$

$A = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} -5 & 4 \\ -2 & 2 \end{bmatrix}$

$|A| = -2(-1)-3 = -1$ $\quad |B| = 3-1(-2) = 5$ $\quad |C| = -5\times 2 - 4(-2) = -10+8 = -2$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix}^T = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

Let $X = A^{-1}+B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix}$

$|X| = 12-4(-1) = 16$

<u>Short Cut</u>

$$\begin{aligned} |D| &= |(A^{-1}+B)^4 C^{-7}| \\ &= |(A^{-1}+B)^4| \times |C^{-7}| \\ &= |(A^{-1}+B)^4|\,|(C^{-1})^7| \\ &= |X|^4 \left(\frac{1}{|C|}\right)^7 \\ &= 16^4 \left(\frac{1}{-2}\right)^7 \\ &= -512 \end{aligned}$$

<u>Long way</u>

$$\begin{aligned} X^4 &= \begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 16-4 & -4-3 \\ 16+12 & -4+9 \end{bmatrix}\begin{bmatrix} 12 & -7 \\ 28 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 12 & -7 \\ 28 & 5 \end{bmatrix}\begin{bmatrix} 12 & -7 \\ 28 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 144-196 & -84-35 \\ 336+140 & -196+25 \end{bmatrix} \\ &= \begin{bmatrix} -52 & -119 \\ 476 & -171 \end{bmatrix} \end{aligned}$$

$|x| = -52(-171)-476(-119) = 65536$

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} 2 & +2 \\ -4 & -5 \end{bmatrix}^T = \frac{1}{-2}\begin{bmatrix} 2 & -4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2.5 \end{bmatrix}$$

$|C^{-1}| = -1\times 2.5 - 2(-1) = -0.5$

$$\begin{aligned} |D| &= |(A^{-1}+B)^4 C^{-7}| \\ &= |(A^{-1}+B)^4| \times |C^{-7}| \\ &= |(A^{-1}+B)^4|\,|(C^{-1})^7| \\ &= 65536 \times (-0.5)^7 \\ &= -512 \end{aligned}$$

Can check every step using GC

Type $\det\left[([A]^{-1}+[B])^4([C]^{-1})^7\right]$ into GC,

GC gives $-511.9999....$ as answer

2.(a)

$$|A-\lambda I| = \begin{vmatrix} -5-\lambda & -8 \\ 4 & 7-\lambda \end{vmatrix} = (-5-\lambda)(7-\lambda) - 4(-8)$$

$$= -35+5\lambda-7\lambda+\lambda^2+32$$

$$= \lambda^2-2\lambda-3$$

$$= (\lambda-3)(\lambda+1)$$

$$= 0$$

$\lambda_1 = 3$, $\lambda_2 = -1$ (smallest absolute value)

For $\lambda_2 = -1$

$$\begin{bmatrix} -5+1 & -8 \\ 4 & 7+1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -8 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4u_1 - 8u_2 = 0$ $\times(-1)$

$4u_1 + 8u_2 = 0$

$4u_1 = -8u_2$

$u_1 = -2u_2$

when $u_2 = 1, u_1 = -2$

$$u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Scale to Euclidean norm = 1

$$u = \frac{1}{\sqrt{(-2)^2+1^2}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(b)(i) $P_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad P_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$q = Gp$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}_{2\times2}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_{2\times1}$$

When P is at P_1, Q is at q_1

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$2g_{11} + g_{12} = 1$ (1)

$2g_{21} + g_{22} = 2$ (2)

When P is at P_2, Q is at q_1

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$g_{11} + 0g_{12} = 1$

$g_{11} = 1$

$g_{21} + 0g_{22} = 2$

$g_{21} = 2$

Sub into eqn (1) & (2)

$2(1) + g_{12} = 1$

$g_{12} = -1$

$2(2) + g_{22} = 2$

$g_{22} = -2$

Hence $G = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ Use GC to check

$|G| = -2 - 2(-1)$

$= 0$ (not invertible) $\longrightarrow$ you can also see that G^{-1} using GC gives an error

Since $|G| = 0$, G is not full rank = 2

$\therefore$ rank(G) < 2 = 1

$\begin{matrix} 1 & -1 \\ 2 & -2 \end{matrix}$ $\times 2 \Rightarrow$ linear combination of one another

(ii)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$P_1 - P_2 = 0$ $\times 2$

$2P_1 - 2P_2 = 0$

$P_1 = P_2$

$\therefore$ P can be anywhere as long as P_1 is equal to P_2 for any arbitrary constant.

$\longrightarrow$ Use GC to sub in any arbitrary constant to check

3.(a) 3^{rd} order NIP $\longrightarrow$ need 4 points

|dist to 7|	6	5	3	1	2
t_i [s]	1	2	4	6	9
$x(t_i)$ [m]	2	4	10	18	33

choose $t_i = 2, 4, 6, 9$

Based on ascending

t_i	$x(t_i)$	First	Second	Third
2	4	$\frac{10-4}{4-2} = 3$	$\frac{4-3}{6-2} = \frac{1}{4}$	$\frac{0.2-0.25}{9-2} = -\frac{1}{140}$
4	10	$\frac{18-10}{6-4} = 4$	$\frac{5-4}{9-4} = \frac{1}{5}$	
6	18	$\frac{33-18}{9-6} = 5$		
9	33			

$$x_3(t_i) = 4 + 3(t-2) + \frac{1}{4}(t-2)(t-4) - \frac{1}{140}(t-2)(t-4)(t-6)$$

$$x_3(7) = 4 + 3(5) + \frac{1}{4}(5)(3) - \frac{1}{140}(5)(3)(1)$$
$$= \frac{317}{14}$$
$$= 22.64\text{ m (location)}$$

$$x_3(t_i) = 4 + 3t - 6 + \frac{1}{4}(t^2 - 4t - 2t + 8) - \frac{1}{140}(t^2 - 6t + 8)(t-6)$$
$$= -2 + 3t + \frac{1}{4}(t^2 - 6t + 8) - \frac{1}{140}(t^3 - 12t^2 + 44t - 48)$$

$$x_3'(t_i) = 3 + \frac{1}{4}(2t-6) - \frac{1}{140}(3t^2 - 24t + 44)$$

$$x_3'(7) = 3 + \frac{1}{4}(14-6) - \frac{1}{140}(3(7)^2 - 24(7) + 44)$$
$$= \frac{677}{140}$$
$$= 4.84\text{ m/s (speed)}$$

$$x_3''(t_i) = \frac{1}{4} \times 2 - \frac{1}{140}(6t - 24)$$
$$x_3''(7) = \frac{1}{2} - \frac{9}{70}$$
$$= \frac{13}{35}$$
$$= 0.371\text{ m/s}^2\text{ (acceleration)}$$

Based on distance

t_i	$x(t_i)$	First	Second	Third
6	18	$\frac{33-18}{9-6} = 5$	$\frac{4.6-5}{4-6} = 0.2$	$\frac{\frac{8}{35} - 0.2}{2-6} = -\frac{1}{140}$
9	33	$\frac{10-33}{4-9} = 4.6$	$\frac{3-4.6}{2-9} = \frac{8}{35}$	
4	10	$\frac{4-10}{2-4} = 3$		
2	4			

$$x_3(t_i) = 18 + 5(t-6) + 0.2(t-6)(t-9) - \frac{1}{140}(t-6)(t-9)(t-4)$$
$$x_3(7) = 18 + 5(1) + 0.2(1)(-2) - \frac{1}{140}(1)(-2)(3)$$
$$= \frac{317}{14}$$
$$= 22.64\text{ m (location)}$$

$$x_3(t_i) = 18 + 5t - 30 + 0.2(t^2 - 9t - 6t + 54) - \frac{1}{140}(t^2 - 15t + 54)(t-4)$$
$$= -12 + 5t + 0.2t^2 - 3t + 10.8 - \frac{1}{140}(t^3 - 4t^2 - 15t^2 + 60t + 54t - 216)$$
$$= -1.2 + 2t + 0.2t^2 - \frac{1}{140}t^3 + \frac{19}{140}t^2 - \frac{57}{70}t + \frac{54}{35}$$
$$= -\frac{1}{140}t^3 + \frac{47}{140}t^2 + \frac{83}{70}t + \frac{12}{35}$$

$$x_3'(t_i) = -\frac{3}{140}t^2 + \frac{94}{140}t + \frac{83}{70}$$
$$x_3'(7) = -\frac{3}{140}(7)^2 + \frac{94}{140}(7) + \frac{83}{70}$$
$$= \frac{677}{140}$$
$$= 4.84\text{ m/s (speed)}$$

$$x_3''(t_i) = -\frac{6}{140}t + \frac{94}{140}$$
$$x_3''(7) = -\frac{6}{140}(7) + \frac{94}{140}$$
$$= \frac{13}{35}$$
$$= 0.371\text{ m/s}^2\text{ (acceleration)}$$

(b) $$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$$

Simpson's 1/3 (2–4); Trapezoidal (4–5); Simpson's 1/3 (5–7); Simpson's 1/3 (7–10)

x_i [m]	2	3	4	5	6	7	8.5	10
$f(x_i)$ [m]	0	0.95	1.89	1.85	1.80	1.73	0.90	0
$(x_i) f(x_i)$	0	2.85	7.56	9.25	10.8	12.11	7.65	0
	1	4	1			1	4	1
				1	4	1		

Trapezoidal

$$A = \int_a^b f(x)\,dx = \frac{1}{3}(1\times 0 + 4\times 0.95 + 1\times 1.89)$$
$$+ 1 \times \frac{1.89 + 1.85}{2}$$
$$+ \frac{1}{3}(1\times 1.85 + 4\times 1.80 + 1\times 1.73)$$
$$+ \frac{1.5}{3}(1\times 1.73 + 4\times 0.90 + 1\times 0)$$
$$= \frac{569}{300} + 1.87 + \frac{539}{150} + 2.665$$
$$= 10.025\text{ m}^2$$

$$\int_a^b x f(x)\,dx = \frac{1}{3}(1\times 0 + 4\times 2.85 + 1\times 7.56)$$
$$+ 1 \times \frac{7.56 + 9.25}{2}$$
$$+ \frac{1}{3}(1\times 9.25 + 4\times 10.8 + 1\times 12.11)$$
$$+ \frac{1.5}{3}(1\times 12.11 + 4\times 7.65 + 1\times 0)$$
$$= 6.32 + 8.405 + 21.52 + 21.355$$
$$= 57.6\text{ m}^3$$

$$\therefore \bar{x} = \frac{1}{10.025} \times 57.6 = \frac{2304}{401}$$
$$= 5.75\text{ m}$$