CV2019 – MATRIX ALGEBRA & COMPUTATIONAL METHODS
AY21/22 SOLUTION

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Q1.

(a)

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & -3 \\ 2 & 3 & 1 & 6 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{swap\ r1\ and\ r2} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_3+(-0.5)r_1} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & -0.5 & 0.5 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & -0.5 & 0.5 & -1 \end{array}\right] \xrightarrow{r_3+(-0.5)r_2} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -0.5 & 0.5 \end{array}\right]$$

$2x_1 + 3x_2 + x_3 = 6 \rightarrow (1)$

$-x_2 + 3x_3 = -3 \rightarrow (2)$

$-0.5x_3 = 0.5 \rightarrow (3)$

From (3), $x_3 = -1$

Sub x_3 into (2), $x_2 = 1$

Sub x_2 and x_3 into (1), $x_1 = 2$

$$\therefore \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(b)

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & -3 \\ 2 & 3 & 1 & 6 \\ 1 & 1 & 1 & 2 \\ a & b & c & d \end{array}\right] \xrightarrow[swap\ r3\ and\ r4]{swap\ r1\ and\ r4} \left[\begin{array}{ccc|c} a & b & c & d \\ 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_4+(-0.5)r_2} \left[\begin{array}{ccc|c} a & b & c & d \\ 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & -0.5 & 0.5 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} a & b & c & d \\ 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & -0.5 & 0.5 & -1 \end{array}\right] \xrightarrow{r_4+(-0.5)r_3} \left[\begin{array}{ccc|c} a & b & c & d \\ 2 & 3 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -0.5 & 0.5 \end{array}\right]$$

From (a), $x_1 = 2, x_2 = 1, x_3 = -1$

From row 1, $ax_1 + bx_2 + cx_3 = d \rightarrow 2a + b - c = d \rightarrow 2a + b - c - d = 0$

$\therefore$ The condition for $\{a, b, c, d\}$ is $2a + b - c - d = 0$.

(c)

$$C = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, C^{-1} = \begin{bmatrix} -0.5 & 1.5 \\ 1 & -2 \end{bmatrix}, (C^{-1})^2 = \begin{bmatrix} -0.5 & 1.5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -0.5 & 1.5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1.75 & -3.75 \\ -2.5 & 5.5 \end{bmatrix}$$

$$(C^{-1})^4 = (C^{-1})^2(C^{-1})^2 = \begin{bmatrix} 1.75 & -3.75 \\ -2.5 & 5.5 \end{bmatrix}\begin{bmatrix} 1.75 & -3.75 \\ -2.5 & 5.5 \end{bmatrix} = \begin{bmatrix} 12.4375 & -27.1875 \\ -18.125 & 39.625 \end{bmatrix}$$

$$(C^{-1})^5 = (C^{-1})^4(C^{-1}) = \begin{bmatrix} 12.4375 & -27.1875 \\ -18.125 & 39.625 \end{bmatrix}\begin{bmatrix} -0.5 & 1.5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -33.40625 & 73.03125 \\ 48.6875 & -106.4375 \end{bmatrix}$$

$$D = (C^{-1})^5A + (C^{-1})^5B = \begin{bmatrix} 52.0625 & 477.8125 \\ -75.875 & -696.375 \end{bmatrix}$$

$$\det(D) = \begin{vmatrix} 52.0625 & 477.8125 \\ -75.875 & -696.375 \end{vmatrix} = -1$$

Q2.

(a)

$A - \lambda I = \begin{bmatrix} -1-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}, |A - \lambda I| = (-1-\lambda)(-1-\lambda) - (3)(3) = 0$

$\lambda^2 + 2\lambda - 8 = 0 \rightarrow (\lambda - 2)(\lambda + 4) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = -4$

For $\lambda_2 = -4$,

$$\begin{bmatrix} -1-(-4) & 3 \\ 3 & -1-(-4) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3u_1 + 3u_2 = 0 \rightarrow u_1 = -u_2$

$\therefore \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{(1)^2 + (-1)^2}} u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$

(b)(i) $\{p_1, p_2, p_3\}$ are linearly independent. This is because the vectors do not have components which are matched by a linear combination of the rest of the group, i.e., $ap_1 + bp_2 + cp_3$ cannot be equal to zero for any combination of a, b, and c.

(ii) q_1, q_2 and q_3 are collinear, which means B is not invertible.

(iii)

Let $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$q_3 = Bp_3 \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

$0.5b_1 + b_2 = 0 \rightarrow 0.5b_1 = -b_2, 0.5b_3 + b_4 = 0 \rightarrow 0.5b_3 = -b_4$

$q_1 = Bp_1 \rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$b_1 - 0.5b_2 = -2 \rightarrow (1), b_3 - 0.5b_4 = 1 \rightarrow (2)$

Sub b_1 into (3),

$-2b_2 - 0.5b_2 = -2 \rightarrow b_2 = 0.8, b_1 = -1.6$

Sub b_3 into (4),

$-2b_4 - 0.5b_4 = 1 \rightarrow b_4 = -0.4, b_3 = 0.8$

$\therefore B = \begin{bmatrix} -1.6 & 0.8 \\ 0.8 & -0.4 \end{bmatrix}$

Q3.

(a) $P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$

$$P(x) = \frac{(x-h)(x-2h)}{(0-h)(0-2h)} y_1 + \frac{(x-0)(x-2h)}{(h-0)(h-2h)} y_2 + \frac{(x-0)(x-h)}{(2h-0)(2h-h)} y_3$$

$$P(x) = \frac{x^2-3hx+2h^2}{2h^2} y_1 - \frac{x^2-2hx}{h^2} y_2 + \frac{x^2-hx}{2h^2} y_3$$

$$P(x) = \frac{1}{2h^2}\left[(x^2 - 3hx + 2h^2)y_1 - 2(x^2 - 2hx)y_2 + (x^2 - hx)y_3\right]$$

$$I = \int_0^{2h} \frac{1}{2h^2}\left[(x^2 - 3hx + 2h^2)y_1 - 2(x^2 - 2hx)y_2 + (x^2 - hx)y_3\right]$$

$$I = \frac{1}{2h^2}\left[\left(\frac{x^3}{3} - \frac{3hx^2}{2} + 2h^2x\right) y_1 - 2\left(\frac{x^3}{3} - hx^2\right) y_2 + \left(\frac{x^3}{3} - \frac{hx^2}{2}\right) y_3\right]_0^{2h}$$

$$I = \frac{1}{2h^2}\left[\left(\frac{8h^3}{3} - \frac{12h^3}{2} + 4h^3\right) y_1 - 2\left(\frac{8h^3}{3} - 4h^3\right) y_2 + \left(\frac{8h^3}{3} - \frac{4h^3}{2}\right) y_3\right]$$

$$I = \frac{1}{2h^2}\left[\left(\frac{2h^3}{3}\right) y_1 + \left(\frac{8h^3}{3}\right) y_2 + \left(\frac{2h^3}{3}\right) y_3\right] = \left(\frac{h}{3}\right)(y_1 + 4y_2 + y_3)$$

(b)

$x\ (m)$	0	2	3	4	5	6	8	10	12
$\rho\ (g/cm^3)$	2.00	1.95	1.89	1.85	1.80	1.73	1.60	1.51	1.49
$\rho\ (kg/m^3)$	2000	1950	1890	1850	1800	1730	1600	1510	1490
$A\ (cm^2)$	31.4	33.2	34.1	35.3	35.8	36.6	37.6	39.0	41.1
$A * 10^{-4} (m^2)$	31.4	33.2	34.1	35.3	35.8	36.6	37.6	39.0	41.1
$\rho * A\ (kg/m)$	6.2800	6.4740	6.4449	6.5305	6.4440	6.3318	6.0160	5.8890	6.1239
$area\ (kg)$	12.754		12.928		12.879		36.128		

The workings are as follows:

x_i = 0 to 2 (Trapezoidal Rule):

$$area = \frac{1}{2}(2)(6.28 + 6.4740) = 12.754kg$$

x_i = 2 to 4 (Simpson's 1/3 Rule):

$$area = \frac{1(1)}{3}(6.4740 + 4(6.4449) + 6.5305) = 12.928kg$$

x_i = 4 to 6 (Simpson's 1/3 Rule):

$$area = \frac{1(1)}{3}(6.5305 + 4(6.4440) + 6.3318) = 12.879kg$$

x_i = 6 to 12 (Simpson's 3/8 Rule):

$$area = \frac{3(2)}{8}(6.3318 + 3(6.0160) + 3(5.8890) + 6.1239) = 36.128kg$$

$$sum\ of\ area = 12.754 + 12.928 + 12.879 + 36.128 = 74.689kg$$

∴ Mass of rod is 74.689kg.

Q4.

(a) $Let\ v = \frac{du}{dt}, \frac{dv}{dt} = \frac{d^2u}{dt^2}$

$\frac{d^2u}{dt^2} - 0.2\frac{du}{dt} + 10u = 0 \rightarrow \frac{dv}{dt} - 0.2v + 10u = 0 \rightarrow \frac{dv}{dt} = 0.2v - 10u$

t	u	v	$\frac{du}{dt}$	$\frac{dv}{dt}$	u_m	v_m	$\frac{du}{dt}_m$	$\frac{dv}{dt}_m$
0	0.2	1	1 (a)	-1.8 (b)	0.25 (c)	0.91 (d)	0.91 (e)	-2.318 (f)
0.1	0.291 (g)	0.7682 (h)	0.7682	-2.7564	0.3294	0.6304	0.6304	-3.1679
0.2	0.3540	0.4514						

The relevant workings are as follows:

(a): $\frac{du}{dt} = v = 1$

(b): $\frac{dv}{dt} = 0.2v - 10u = 0.2(1) - 10(0.2) = -1.8$

(c): $u_m = u_i + 0.5K_0h = 0.2 + 0.5(1)(0.1) = 0.25$

(d): $v_m = v_i + 0.5K_0h = 1 + 0.5(-1.8)(0.1) = 0.91$

(e): $\frac{du}{dt}_m = v_m = 0.91$

(f) $\frac{dv}{dt}_m = 0.2v - 10u = 0.2(0.91) - 10(0.25) = -2.318$

(g) $u_{i+1} = u_i + \frac{du}{dt}_m h = 0.2 + (0.91)(0.1) = 0.291$

(g) $v_{i+1} = v_i + \frac{dv}{dt}_m h = 1 + (-2.318)(0.1) = 0.7682$

Repeat for the next iteration gives:

$u(0.2) = 0.3540m, v(0.2) = 0.4514m/s$

(b) For $T_{0,1}$:

$100 + 25 + 2T_{1,1} - 4T_{0,1} = 0 \rightarrow (1)$

For $T_{1,1}$:

$T_{0,1} + 75 + T_{2,1} + 25 + 4T_{1,1} = 0 \rightarrow T_{0,1} - 4T_{1,1} + T_{2,1} = -100 \rightarrow (2)$

For $T_{2,1}$:

$T_{1,1} + 50 + 25 + 25 - 4T_{2,1} = 0 \rightarrow T_{1,1} - 4T_{2,1} = -100 \rightarrow (3)$

Solving (1), (2), and (3), gives $T_{0,1} = 55.288°C, T_{1,1} = 48.077°C, T_{2,1} = 37.019°C$.

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NOTE:

Do reach out to me at KEAL0001@e.ntu.edu.sg if you have any queries regarding any of my submitted workings. Feel free to leave an email to ask any questions covered in the curriculum, will be glad to help!

DISCLAIMER:

You are advised to take my solutions as a **guide**, rather than an absolute answer to the questions.