CV2019 – MATRIX ALGEBRA & COMPUTATIONAL METHODS
AY20/21 SOLUTION

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Q1.

(a)

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & -2 \\ -1 & 1 & -1 & -2 \\ 5 & 1 & 5 & 4 \end{array}\right] \xrightarrow[r_3+(2.5)r_1]{r_2+(0.5)r_1} \left[\begin{array}{ccc|c} 2 & 4 & 2 & -2 \\ 0 & 3 & 0 & -3 \\ 0 & -9 & -5 & 9 \end{array}\right] \xrightarrow{r_3+(3)r_2} \left[\begin{array}{ccc|c} 2 & 4 & 2 & -2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

$2x_1 + 4x_2 + 2x_3 = -2 \rightarrow (1)$

$3x_2 = -3 \rightarrow (2)$

$-5x_3 = 0 \rightarrow (3)$

From $(2), x_2 = -1.$

$2x_1 + 4(-1) + 2(0) = -2 \rightarrow x_1 = 1 - x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -3 & 1 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

From row 4, $x_4 = d$

From row 3, $-x_3 + x_4 = c \rightarrow x_3 = d - c$

From row 5, $fx_3 + 2x_4 = e \rightarrow fx_3 + 2d = e$

$f(d - c) + 2d = e \rightarrow e + (c \times d) - \big(d(f + 2)\big) = 0$

(c)(i) $|A| = (-1)(2) - (1)(2) = -4$

$$A^{-1} = \frac{1}{-4} \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.25 & 0.25 \end{bmatrix}$$

(ii) $\|A\|_2 = \sqrt{(-1)^2 + (2)^2 + (1)^2 + (2)^2} = \sqrt{10}$

$\|A^{-1}\|_2 = \sqrt{(-0.5)^2 + (0.5)^2 + (0.25)^2 + (0.25)^2} = \sqrt{0.625}$

$K_2(A) = \|A\|_2 * \|A^{-1}\|_2 = \sqrt{10}\sqrt{0.625} = 2.5$

(iii) Let x_i/y_i be the initial coordinates and x_n/y_n be the final coordinates.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}, A^{-1} = \begin{bmatrix} -0.5 & 0.5 \\ 0.25 & 0.25 \end{bmatrix}$$

$\|x_n - x_i\| = \|A^{-1}\|\|y_n - y_i\| \leq \|A^{-1}\|(10)$

$\|A^{-1}\|(10) = \sqrt{0.625}(10) = 7.9057km$

Therefore, the region should be less than 7.91km.

Q2.

(a)(i)

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (3-\lambda)(1-\lambda) - (-1)(-1) = 3 - \lambda - 3\lambda + \lambda^2 - 1 = \lambda^2 - 4\lambda + 2$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = 2 \pm \sqrt{2}, \lambda_1 = 2+\sqrt{2}, \lambda_2 = 2-\sqrt{2}$$

$$For\ \lambda_2 = 2-\sqrt{2},$$

$$\begin{bmatrix} 3-(2-\sqrt{2}) & -1 \\ -1 & 1-(2-\sqrt{2}) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+\sqrt{2} & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1+\sqrt{2})u_1 - u_2 = 0$$

$$u_1 = \frac{1}{1+\sqrt{2}}\frac{(1-\sqrt{2})}{(1-\sqrt{2})}u_2 \rightarrow u_1 = (-1+\sqrt{2})u_2$$

$$\therefore \boldsymbol{u} = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} u_2$$

(ii)

$$M^{-1} = \begin{bmatrix} \frac{1}{1000} & 0 \\ 0 & \frac{1}{1000} \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} \frac{1}{1000} & 0 \\ 0 & \frac{1}{1000} \end{bmatrix}\begin{bmatrix} 3000 & -1000 \\ -1000 & 1000 \end{bmatrix} \times 10^3 = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \times 10^3$$

$$|A-\lambda I| = \begin{vmatrix} 3000-\lambda & -1000 \\ -1000 & 1000-\lambda \end{vmatrix} = \lambda^2 - (4\times 10^3)\lambda + 2\times 10^6$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = 2000 \pm 1000\sqrt{2}, \lambda_1 = 2000+1000\sqrt{2}, \lambda_2 = 2000-1000\sqrt{2}$$

$$\omega_1 = \sqrt{\lambda_1} = 58.431, \omega_2 = \sqrt{\lambda_2} = 24.203$$

$$f_1 = \frac{\omega_1}{2\pi} = 9.300Hz, f_2 = \frac{\omega_2}{2\pi} = 3.852Hz$$

∴ Lowest natural frequency is 3.852Hz.

(b)(i)

$$P = Q^TQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(Q) = \frac{1}{\sqrt{2}}\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} - \frac{1}{\sqrt{2}}\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} + 0 = \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\left(-\frac{1}{2\sqrt{2}}\right) = 1$$

(ii)

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, U = Q$$

Since determinant of Q is 1, and Q^TQ is an identity matrix,

$$U^{-1} = Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = UDU^{-1} = \begin{bmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B = (A - 2I)^3 = \left(\begin{bmatrix} 1.5 & -0.5 & 0 \\ -0.5 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right)^3$$

$$B = \left(\begin{bmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^3 = \begin{bmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q3.

(a)

Time, s (s)	0	3	6	8	10	12	15	18	20
Speed, v (kph)	0	20	47	66	85	102	124	141	150
Speed, v (m/s)	0	5.556	13.056	18.333	23.611	28.333	34.444	39.167	41.667
Area (m)	35.28		125.416			205.276		80.834	

Use Time (s) as x_i and Speed (m/s) as $f(x_i)$ to determine Area (m). The workings are as follows:

x_i = 0 to 6 (Simpson's 1/3 Rule):

$$area = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = \frac{3}{3}(0 + 4(5.556) + 13.056) = 35.28$$

x_i = 6 to 12 (Simpson's 3/8 Rule):

$$area = \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

$$area = \frac{3(2)}{8}(13.056 + 3(18.333) + 3(23.611) + 28.333) = 125.416$$

x_i = 12 to 18 (Simpson's 1/3 Rule):

$$area = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = \frac{3}{3}(0 + 5.556 + 13.056) = 35.28$$

x_i = 18 to 20 (Trapezoidal Rule):

$$area = \frac{h}{2}\left(f(x_0) + f(x_1)\right) = \frac{2}{2}(39.167 + 41.667) = 80.834$$

$$sum\ of\ area = 35.28 + 125.416 + 205.276 + 80.834 = 446.806m$$

∴ Distance travelled is 446.806m.

(b)(i) Choose the data pairs at time t = 6, 8, 10 and 12.

i	x_i	$f(x_i)$	First	Second	Third
0	6	47	9.5	0	-0.041667
1	8	66	9.5	-0.25	
2	10	85	8.5		
3	12	102			

$v(t) = 47 + 9.5(t-6) - 0.041667(t-6)(t-8)(t-10)$

(ii) $v(9) = 47 + 9.5(9-6) - 0.041667(9-6)(9-8)(9-10) \;= 75.625kph \; or \; 21m/s$

(iii) $v(t) = 47 + 9.5t - 57 - 0.041667(t^3 - 24t^2 + 188t - 480)$

$v'(t) = 9.5 - 0.041667(3t^2 - 48t + 188)$

$v'(10) = 9.5 - 0.041667(3(10)^2 - 48(10) + 188) = 13.333\frac{km/h}{s} = 2.55m/s^2$

(c) $acceleration = f''(u_i) = f'(v_i)$

$f'(v_i) = \frac{f(v_{i+1}) - f(v_{i-1})}{2h} = \frac{28.333 - 18.333}{2(2)} = 2.50m/s^2$

Q4.

(a)(i) Given that $y = \frac{x}{\ln x}, \frac{dy}{dx} = \frac{1}{\ln x} - \frac{1}{(\ln x)^2}$, sub y and $\frac{dy}{dx}$ into D.E:

$x^2\left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right) = x\left(\frac{x}{\ln x}\right) - \left(\frac{x}{\ln x}\right)^2$, expanding both sides give:

$\frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2} = \frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2}$

Since LHS = RHS, the function $y = \frac{x}{\ln x}$ is an analytical solution for the D.E.

(ii) From D.E: $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$

$y_{e+0.2} = y_e + f(x_e, y_e)\Delta x$

$f(x_e, y_e) = \left.\frac{dy}{dx}\right|_e = \frac{e^2 - e^2}{e^2} = 0$

$y_{e+0.2} = y_e + 0(0.2) = e + 0 = 2.7183$

$y_{e+0.4} = y_{e+0.2} + f(x_{e+0.2}, y_{e+0.2})\Delta x$

$f(x_{e+0.2}, y_{e+0.2}) = \left.\frac{dy}{dx}\right|_{e+0.2} = \frac{(e+0.2)(e) - e^2}{e^2} = 0.063837$

$y_{e+0.4} = y_e + 0.063837(0.2) = 2.7311$

(iii) From analytical solution, $y = \frac{x}{\ln x}, y_{e+0.4} = \frac{e+0.4}{\ln(e+0.4)} = 2.7419$

$absolute\ percentage\ error = \frac{2.7419 - 2.7311}{2.7311} \times 100\% = 0.395\%$

(b) $y = 10e^{-kt}\cos\omega t = 5$

Let $f(x) = 10e^{-0.8t}\cos 5t - 5$

$x_{i+1} = x_i - \frac{f(x_i)(x_{i-i} - x_i)}{f(x_{i-1}) - f(x_i)}$

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	x_{i+1}	(x_{i+1})	ε_a (%)
0	0.2	-0.3958	0.3	-4.4436	0.19022	-0.011977	-0.23955
1	0.3	-4.4436	0.19022	-0.011977	0.18992	-0.000453	-0.00905

After two iterations, $t = 0.18992 \approx 0.190$

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NOTE:

Do reach out to me at KEAL0001@e.ntu.edu.sg if you have any queries regarding any of my submitted workings. Feel free to leave an email to ask any questions covered in the curriculum, will be glad to help!

DISCLAIMER:

You are advised to take my solutions as a **guide**, rather than an absolute answer to the questions.